ON STRONG APPROXIMATION APPLIED TO
POST-WIDDER OPERATORS

Lucyna Rempulska and Mariola Skorupka
(Poznań University of Technology, Poland)

Received Dec. 29, 2005

Abstract

We introduce modified Post-Widder operators in polynomial weighted spaces of differentiable functions and we study strong approximation for them.

Key words
Post-Widder operator, strong approximation, degree of approximation

AMS(2000) subject classification 41A36, 41A25

1 Introduction

1.1. Approximation properties of Post-Widder operators

\[ P_n(f; x) := \int_0^\infty p_n(x, t) f(t)dt, \quad x \in I, \ n \in \mathbb{N}, \]  \hspace{1cm} (1)

\[ p_n(x, t) := \left( \frac{n}{x} \right)^n \frac{t^{n-1}}{(n-1)!} \exp \left( -\frac{nt}{x} \right), \] \hspace{1cm} (2)

\[ I = (0, \infty), \ \mathbb{N} = \{1, 2, \ldots\}, \] for real-valued functions \( f \) continuous and bounded on \( I \) were examined in [2] (Chapter 9).

It is known that \( P_n \) are well defined also for \( f_r(x) = x^r, \ r \in \mathbb{N}, \) and

\[ P_n(1; x) = 1, \quad P_n(t; x) = x \quad \text{for} \quad x \in I, \ n \in \mathbb{N}. \] \hspace{1cm} (3)

Generally, for \( n, r \in \mathbb{N} \) and \( x \in I, \) we have

\[ P_n(t^r; x) = \frac{n(n+1) \cdots (n+r-1)x^r}{n^r}, \hspace{1cm} \] \hspace{1cm} (4)
which implies that
\[ P_n ((t - x)^2; x) = \frac{x^2}{n} \]
and for every \( 2 \leq r \in N \) there exists a positive constant \( M_1(r) \) depending only on \( r \) such that
\[ P_n ((t - x)^{2r}; x) \leq M_1(r) n^{-r} x^{2r} \]
for \( x \in I \) and \( n \in N^2 \). Moreover, from theorems given in [2], Chapter 9, we deduce that
\[ |P_n(f; x) - f(x)| \leq M_2 \omega_2 \left( f; \frac{x}{\sqrt{n}} \right), \quad x \in I, \ n \in N \]
for every \( f \) continuous and bounded on \( I \), where \( \omega_2(f) \) is the second modulus of continuity of \( f \) and \( M_2 = \text{const.} > 0 \).

1.2. The problem of strong approximation related with Fourier series was investigated in many papers, e.g. [4].

For example, if \( S_n(f; \cdot) \) and \( \sigma_n(f; \cdot) \) are the \( n \)-th sum and \((C,1)\)-mean of trigonometric Fourier series of \( f \), respectively, i.e.
\[ \sigma_n(f; x) := \frac{1}{n+1} \sum_{k=0}^{n} S_k(f; x), \]
then
\[ \sigma_n(f; x) - f(x) = \frac{1}{n+1} \sum_{k=0}^{n} (S_k(f; x) - f(x)), \quad x \in R, \ n \in N_0 = N \cup \{0\}. \]

The strong approximation of \( f \) by \((C,1)\)-means of Fourier series is connected with the following strong differences
\[ H^q_n(f; x) := \left( \frac{1}{n+1} \sum_{k=0}^{n} |S_k(f; x) - f(x)|^q \right)^{1/q}, \quad x \in R, \ n \in N_0, \]
where \( q > 0 \) is a fixed number. It is easily verified that
\[ |\sigma_n(f; x) - f(x)| \leq H^1_n(f; x) \]
and
\[ H^p_n(f; x) \leq H^q_n(f; x), \quad 0 < p < q < \infty \]
for \( x \in R \) and \( n \in N_0 \). These inequalities show that the consideration of strong differences \( H^q_n(f) \) for Fourier series of \( f \) is useful.

1.3. The purpose of this note is to investigate the strong approximation of functions by their Post-Widder operators. We shall define certain modified Post-Widder operators in polynomial weighted space of \( r \) times differentiable functions and we shall examine their strong differences.