Approximations in Functional Analysis

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Abstract

Starting from a normed real vectorspace \((X, \| \cdot \|)\), we introduce a uniform approach structure on \(X\) (resp. \(X'\)) the topological and cop-metric coreflection of which are the weak topology \(\sigma (X, X')\) and the metric \(d_{\| \cdot \|}\) (resp. the weak* topology \(\sigma (X', X)\) and the metric \(d_{\| \cdot \|'}\)). We investigate the properties of these introduced approach structures, obtaining quantitative results which imply their classical qualitative counterparts.

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1 Introduction

In the study of normed linear spaces and their duality, the so-called weak and weak* topologies play an important role. It is out of necessity, unfortunately, that we loose the canonical numerical information we dispose of in the given normed space and its normed dual when considering these weak and weak* structures. The reason for this is that they are formed out of the original normed objects by taking initial liftings, which of course can be constructed on the topological, but fail to exist on the normed or metric level. The solution we investigate in this paper is stepping outside the topological frame into the broader numerical framework of approach spaces as introduced in (R. Lowen [8]), in which these initial liftings can be performed in a way compatible with the well studied topological objects. It is our aim to present a study of the basic properties of the introduced structures and to show that many classical results now can be obtained as simple corollaries.

2 Preliminaries

In (R. Lowen [8]) "approach spaces" were introduced to answer the question what numerical information can be preserved, while maintaining compatibility with the underlying topologies, when constructing products, quotients and coproducts of metric spaces, since the metric structure is extremely unstable with respect to the mentioned constructions. It is the aim of this paragraph to give an account of some basic definitions and theorems concerning approach spaces, but we refer to (E. Lowen and R. Lowen [5], R. Lowen and K. Robeys [6] and R. Lowen [8], [10]) for more information. For any background material on categorical notions, we refer to (J. Adámek, H. Herrlich and G. Strecker [1]). We will use \(\| \cdot \|_E\) (resp. \(d_E\) and \(T_E\)) to denote the Euclidean norm (resp. the Euclidean metric and the Euclidean topology) on \(\mathbb{R}\) and when working in \([0, \infty)\), we adopt the convention that \(\infty \cdot 0 = 0 \cdot \infty = 0\). In order not to overload notations, we will, for an arbitrary set \(X\), write \(2^X\) (resp. \(2_0^X\), \(2(X)\), \(2_0(X)\)) for the set of all (resp. all non-empty, all finite, all non-empty finite) subsets of \(X\).

To begin with, we define the concept of an "approach distance" on \(X\), which can be seen as a numerical counterpart to a topological closure operator, indicating "how far a given point of \(X\) is away from being an adherence point of a given subset of \(X\)."

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Definition 2.1: Consider a function \( \delta : X \times 2^X \to [0, \infty] \). For every \( A \subseteq X \) and every \( \varepsilon \in \mathbb{R}^+ \) we write \( A^{(\varepsilon)} = \{ x \in X | \delta(x, A) \leq \varepsilon \} \). Then \( \delta \) is called an "approach distance" (or shortly a "distance") on \( X \) if it satisfies following conditions:

\begin{enumerate}
  \item[(D1)] \( \forall x \in X : \delta(x, \{ x \}) = 0 \),
  \item[(D2)] \( \forall x \in X : \delta(x, \emptyset) = \infty \),
  \item[(D3)] \( \forall x \in X, \forall A, B \subseteq X : \delta(x, A \cup B) = \delta(x, A) \wedge \delta(x, B) \),
  \item[(D4)] \( \forall x \in X, \forall A \subseteq 2^X, \forall \varepsilon \in \mathbb{R}^+ : \delta(x, A) \leq \delta(x, A^{(\varepsilon)}) + \varepsilon \).
\end{enumerate}

To continue, we give the definition of an "approach system" on \( X \), which is to be interpreted as a quantified counterpart to a topological neighbourhood system.

Definition 2.2: A collection of order-theoretic ideals \( \mathcal{A} = (A(x))_{x \in X} \) in \([0, \infty]^X\) is called an "approach system" on \( X \) if it satisfies the following conditions for every \( x \in X \):

\begin{enumerate}
  \item[(A1)] \( \forall \varphi \in A(x) : \varphi(x) = 0 \),
  \item[(A2)] \( \forall \varphi \in [0, \infty)^X : \forall \varphi, \omega \in [0, \infty)^X : \exists \varphi^{\omega, \varepsilon} \in \mathcal{A}(x) : \varphi \wedge \omega \leq \varphi^{\omega, \varepsilon} + \varepsilon \Rightarrow \varphi \in \mathcal{A}(x) \),
  \item[(A3)] \( \forall \varphi \in A(x), \forall \omega \in [0, \infty)^X : \exists (\varphi_x)_{x \in X} \in \prod_{x \in X} A(x) : \forall z, y \in X : \varphi(y) \wedge \omega \leq \varphi_z(z) + \varphi_z(y) \).
\end{enumerate}

To see this resemblance, note that for any \( x, y \in X \) and any \( \varphi \in A(x), \varphi(y) \) has to be seen as the "distance from \( x \) to \( y \) according to \( \varphi \). A family of order-theoretic ideal bases which only satisfies conditions (A1) and (A3) is called an "approach basis" on \( X \) and it was shown in [R. Lowen [8], [10]] that each approach basis on \( X \) generates an approach system on \( X \) by saturating it with respect to condition (A2).

The following result, which can be found in [R. Lowen [8], [10]] shows that approach distances and approach systems are equivalent concepts, which justifies the definition immediately following it.

Theorem 2.3: 1. If \( \delta \) is a distance on \( X \), then

\[ A_\delta(x) = \{ \varphi \in [0, \infty]^X | \forall A \subseteq 2^X : \inf_{a \in A} \varphi(a) \leq \delta(x, A) \} \quad x \in X, \]

defines an approach system on \( X \).

2. If \( \mathcal{A} = (A(x))_{x \in X} \) is an approach system on \( X \), then

\[ \delta_{\mathcal{A}}(x, A) = \sup_{\varphi \in A(x)} \inf_{a \in A} \varphi(a) \quad x \in X, A \subseteq 2^X, \]

defines a distance on \( X \).

The transitions described above are one-to-one, i.e. for every approach distance \( \delta \) on \( X \), \( \delta_{\mathcal{A}} = \delta \) and for every approach system \( \mathcal{A} \) on \( X \), the equality \( A_{\mathcal{A}} = A \) holds. (The transition formula \( \delta \) remains valid when the approach system \( \mathcal{A} \) is replaced by an approach basis generating it.)

Definition 2.4: An "approach space" is a pair \((X, \mathfrak{S})\) where \( X \) is an arbitrary set and \( \mathfrak{S} \) is an approach distance or an approach system on \( X \).

In [R. Lowen [10]] many other equivalent formalisms to describe approach spaces are given, e.g. "approach limits". In this brief overview we will only indicate how "approach limits" can be obtained from approach distances or approach systems. Throughout the text, we will use \( \mathcal{F}(X) \) (resp. \( \mathcal{U}(X) \)) to denote the set of all filters (resp. all ultrafilters) on \( X \) and for any \( \mathcal{F} \in \mathcal{F}(X) \) we write

\[ \text{sec} \mathcal{F} = \bigcup_{u \in \mathcal{U}(X), u \supseteq \mathcal{F}} u. \]

If \( \mathcal{B} \) is a filterbasis on \( X \), we will write \( \text{stack} \mathcal{B} \) for the filter \( \{ F \subseteq X | \exists B \subseteq \mathcal{B} : B \subseteq F \} \) on \( X \) generated by \( \mathcal{B} \), and if \( (x_\alpha)_\alpha \) is a net in \( X \) based on a directed set \( (D, \preceq) \), \( (x_\alpha)_\alpha \) will stand for the filter on \( X \)