INTERPOLATION FUNCTORS IN WEAK-TYPE INTERPOLATION

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ABSTRACT. For interpolation in the diagonal case, i.e. with respect to the two couples \((X, X)\) and \((Y, Y)\), there exists a natural relation between weak-type and strong-type interpolation. Indeed, weak-type interpolation is related to the "M-couples" \((AX, MX)\) and \((AY, MY)\) of the Lorentz spaces of \(X\) and \(Y\). Since \(AZ \subseteq MZ\) for any space \(Z\), any weak-type interpolation space also has the (strong-type) interpolation property for the "A-couples" \((AX, AX)\) and \((AY, AY)\). In this paper a scale \(c, c > 0\), of interpolation functors with respect to the \(A\)-couples is introduced such that all generated interpolation spaces (also) have the weak-type interpolation property. Moreover, we will show that a space is a weak interpolation space if and only if it is generated by one of these functors.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper let \(X, Y, X_i, Y_i\) denote rearrangement invariant Banach function spaces (r.i. spaces) in the sense of Luxemburg-Zaanen [1]. In particular,

\[ \text{if } |g| \leq |f| \text{ a.e. and } f \in X, \text{ then } g \in X \text{ and } \|g\|_X \leq \|f\|_X. \]

In view of the Luxemburg representation theorem there is no loss of generality in confining ourselves to \(((0, \infty), m)\) with Lebesgue measure \(m\) as measure space. For more details see [2], [3] or [4]. If \(X\) is any r.i. space, we denote by \(\varphi_X(t)\) the fundamental function of \(X\), defined by \(\varphi_X(t) = \|\chi_{(0, t)}\|_X\), with \(\chi(0, t)\) being the characteristic function of the interval \((0, t)\). In the sequel, \(\varphi_X\) will always be assumed to be concave (see [5]) and such that \(\varphi_X(0+) = 0\). Associated with a r.i. space \(X\) there are the generalized Lorentz spaces \(AX\), \(MX\), and \(M^*X\), defined by

\[
\Lambda X := \{f; \|f\|_{\Lambda X} := \int_0^\infty f^*(s) \, d\varphi_X(s) < \infty\},
\]

\[
MX := \{f; \|f\|_{MX} := \sup_{t>0} [f^{**}(t) \cdot \varphi_X(t)] < \infty\},
\]

\[
M^*X := \{f; \|f\|_{M^*X} := \sup_{t>0} [f^*(t) \varphi_X(t)] < \infty\},
\]

where \(f^*\) and \(f^{**}\) denote the nonincreasing rearrangement of \(f\) and the averaging function of \(f\), respectively. It is well-known (see for example [3]) that \(AX\) and \(MX\) are r.i. spaces, whereas \(M^*X\) is not a Banach space since \(\|\cdot\|_{M^*X}\) fails to be subadditive. Nevertheless, \(M^*X\) is a complete quasilinear rearrangement invariant function space satisfying property (1). The following inclusions hold

\[ L_1 \cap L_\infty \hookrightarrow \Lambda X \hookrightarrow X \hookrightarrow MX \hookrightarrow M^*X \hookrightarrow L_1 + L_\infty. \]
If, however, $X$ belongs to the class $\mathcal{U}$ introduced by R. Sharpley in [6], then for all $f \in X$ there holds
\[ \|f\|_{M^*X} \leq c(X) \cdot \|f\|_{M^X} \] (see [6] and [7]), yielding that
\[ M^X \cong M^*X \quad \text{if} \quad X \in \mathcal{U}. \] (3)
Recall that any r.i. space $X$ belongs to the class $\mathcal{U}$ if there exist constants $a, \delta \in (0,1)$ and $c > 0$ such that
\[ \frac{\varphi_X(t_2)}{\varphi_X(t_1)} \leq c \cdot \left(\frac{t_2}{t_1}\right)^a \quad \text{if} \quad \frac{t_1}{t_2} \leq \delta. \]
For example, the Lebesgue spaces $L_p$ ($1 \leq p \leq \infty$) are in $\mathcal{U}$ iff $p > 1$. More generally, any r.i. space $X$ is in $\mathcal{U}$ if and only if for the upper fundamental index $\gamma_X$ of $X$, introduced by M. Zippin in [5], there holds $\gamma_X < 1$ or, equivalently, $\alpha_{AX} < 1$, with $\alpha_{AX}$ denoting the upper Boyd index of the space $\Lambda X$ (see [8] and [9]).

In the following we will characterize weak-type interpolation by means of the wider quasilinear space $M^*X$ instead of the Banach space $MX$.

For any interpolation segment
\[ \sigma = [(X_0,Y_0),(X_1,Y_1)] \] (4)
of r.i. spaces we denote by $\mathcal{A}(\sigma)$ the class of all admissible operators $T$ with respect to $\sigma$, that is to say, the class of all linear operators $T$ from $X_0 + X_1$ into $Y_0 + Y_1$ such that $T|_{X_i} \in [X_i,Y_i]$ for $i = 0, 1$. The space $\mathcal{A}(\sigma)$ is a Banach space with $\|T\|_{\sigma} := \max\{\|T\|_{[X_i,Y_i]} : i = 0, 1\}$.

With this notation, a Banach couple $(X,Y)$ of r.i. spaces is said to have the strong-type interpolation property with respect to $\sigma$ (denoted by $(X, Y) \in \text{Int}(\sigma)$) iff $X \hookrightarrow X_0 + X_1$, $Y \hookrightarrow Y_0 + Y_1$ and $T|_X : X \rightarrow Y$ for every $T \in \mathcal{A}(\sigma)$. Observe that by the closed graph theorem the restriction $T|_X$ is continuous on $X$. The particular case $X_0 = Y_0$, $X_1 = Y_1$ and $X = Y$, i.e.
\[ \sigma = \sigma_D := [(X_0,X_0),(X_1,X_1)], \]
will be referred to as the "diagonal case" in strong-type interpolation. In this case we write $X \in \text{Int}(\sigma_D)$ instead of $(X,X) \in \text{Int}(\sigma_D)$. Otherwise we speak of the "off-diagonal case".

In weak-type interpolation we consider the interpolation segment
\[ \overline{\sigma} = [(\Lambda X_0, M^*Y_0), (\Lambda X_1, M^*Y_1)] \] (5)
instead of (4) (see [4]). A Banach couple $(X,Y)$ of r.i. spaces then has the weak-type interpolation property with respect to $\sigma$ (denoted by $(X, Y) \in \text{Weak}(\sigma)$) iff $(X,Y) \in \text{Int}(\overline{\sigma})$.

Notice that whenever $Y_0$ and $Y_1$ belong to the class $\mathcal{U}$ then in view of (3) the segment $\overline{\sigma}$ of (5) is equal to $\overline{\sigma} = [(\Lambda X_0, MY_0), (\Lambda X_1, MY_1)]$. The diagonal case in weak-type interpolation (i.e. $X_0 = Y_0$ and $X_1 = Y_1$) is, in general, an off-diagonal case in strong-type interpolation. Hence, it is not possible to regard weak-type interpolation as a special case of strong-type interpolation. In particular, the (strong-type) methods of e.g. N. Aronszajn/E. Gagliardo, V.I. Ovchinnikov, and Yu.A. Brudnyi/N.Na. Krugliak, respectively, concerning the diagonal case cannot directly be transferred to the weak-type situation. In what follows, we therefore give a more detailed study of the relations between weak- and strong-type interpolation. More precisely, we will consider the weak-type interpolation segment
\[ \overline{\sigma}_D = [(\Lambda X_0, M^*X_0), (\Lambda X_1, M^*X_1)]. \] (6)

In section 2 we compare interpolation with respect to this segment with a particular case of strong-type interpolation, namely with respect to the segment
\[ \sigma_A = [(\Lambda X_0, \Lambda X_0), (\Lambda X_1, \Lambda X_1)]. \] (7)
The method of proof consists in combining the orbit methods of Aronszajn/Gagliardo with a typical weak-type instrument, namely the Calderón operator.