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ARCS IN THE HALL PLANES

Gloria Rinaldi

Abstract. Using a transformation technique for designs introduced in [1], I construct a class of arcs embeddeable in the Hall plane and in the dual of the Hall plane of order \(q^2\) proving also their completeness in the unital of Gruning. Math. Subj. Class.: 51A35 Non Desarguesian affine and projective planes. 51E22 Blocking sets, ovals, k-arcs.

1. Introduction.

Let \(k\) be a positive integer, a \(k\)-arc in a projective plane is a set of \(k\) points no three of which are collinear and it is said to be complete if not properly contained in a \((k + 1)\)-arc. Obviously a \((n + 1)\)-arc in a projective plane of finite order \(n\) is an oval and, for \(n\) even, a \((n + 2)\)-arc is an hyperoval. In what follows let \(q\) be a prime power; if a complete \(k\)-arc in \(PG(2, q^2)\) is not an oval (hyperoval), then:

\[ k \leq q^2 - \frac{q}{4} + \frac{7}{4} \] for \(q\) odd and
\[ k \leq q^2 - q + 1 \] for \(q\) even. These estimations are due to B. Segre, [2], see also [3]. In [4], B. Kestenband constructed a class of \((q^2 - q + 1)\)-arcs in \(PG(2, q^2)\) and several authors proved their completeness, see [5], [6], [7], showing that, for \(q\) even, the estimation of Segre is a good one. In this paper, using a transformation process for incidence structures introduced in [1], I prove that some arcs constructed by Kestenband in \(PG(2, q^2)\) give rise to arcs embeddable in the Hall plane of the same order \(q^2\) and in its dual plane also, furthermore, recalling the results of [8] and [9], I point out some properties of these arcs.

2. Transformation process for projective planes and construction of arcs in the Hall planes.

Let \(\pi = (P, R, I)\) be a projective plane with point-set \(P\), line-set \(R\) and incidence relation \(I\); let \(F\) be a a subset of \(R\) and \(f\) a permutation on \(P\), we say that \(\pi\) is transformable and \(\{F, f\}\) is a transformation system for \(\pi\) if, denoting \(\langle P, Q \rangle\) the line incident with the points \(P\) and \(Q\), the following condition is satisfied:

\[ \langle P, Q \rangle \in F \iff \langle f(P), f(Q) \rangle \in F. \]

If \(\pi\) is transformable, a new point-line incidence relation can be defined as follows:

if \(r \notin F\) then \(PJr \iff PIR\),
if \(r \in F\) then \(PJr \iff f(P)Ir\).

The new incidence structure \(\pi^* = (P, R, J)\) is called the transformed incidence structure of \(\pi\) and, whenever \(\pi\) is a projective plane of finite order, \(\pi^*\) is a projective plane itself and of the same order. This transformation technique is due to P. Quattrocchi and L. A. Rosati, see [1], and finds a lot of applications, see for example [10], [8], [9]. For the readers' convenience I prove the following proposition which is a specific case of proposition 12 of [10].

Proposition 1. Let \(K\) be an arc embedded in a finite projective plane \(\pi\), let \(\{F, f\}\) be a transformation system for \(\pi\) and \(\pi^*\) be the transformed projective plane. If it is \(f(K) = K\), then the set \(K\) is an arc in \(\pi^*\).

Proof: we want to prove that each line of \(\pi^*\) is incident with at most two points of \(K\). Let \(r\) be a line of \(\pi^*\), if \(r \notin F\) it is straightforward; suppose \(r \in F\) and suppose the existence of

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three distinct points of $\mathcal{K} : R, S, T$ such that $R I r, S I r, T I r$, this implies $f(R) I r, f(S) I r, f(T) I r$ with $f(R), f(S), f(T)$ distinct points of $\mathcal{K}$: a contradiction.

We will use this proposition to construct arcs in the Hall planes; first of all let us recall Kestenband’s construction of $(q^2 - q + 1)$-arcs in $PG(2, q^2)$.

Let $H = \begin{pmatrix} a & d & e \\ d^q & b & f \\ e^q & f^q & c \end{pmatrix}$ be a rank three hermitian matrix over $GF(q^2)$, $(H$ hermitian

implies $a, b, c \in GF(q)$); denote the points of $PG(2, q^2)$ by the vectors $X = (x, y, z) \neq (0, 0, 0)$, then the curve $X \cdot X^q = 0$, with $X^q = (x^q, y^q, z^q)$ is a classical unital of order $q$ embedded in $PG(2, q^2)$ (that is a $(q^3 + 1, q + 1, 1)$-design), [11]. We will denote this unital by $\{H\}$. Let $I_3$ denote the identical matrix; if the characteristic polynomial of $H$ is irreducible over $GF(q^2)$, then the $q + 1$ unitals $\{I_3\} \{H - \lambda I_3\}$, $\lambda$ ranging through $GF(q)$, cover the plane and pairwise meet on the same set $\mathcal{K}$. Furthermore $\mathcal{K}$ is an arc containing $q^3 - q + 1$ points, [4]. Let $f$ be the so defined permutation on the point-set of $PG(2, q^2)$: $f(x, y, z) = (x^q, y, z)$, $f(x, y, z) = (x, y, z)$; with these notations we prove the following:

**Proposition 2.** Let $q \geq 5$; it is $f(\mathcal{K}) = \mathcal{K}$ if and only if $d = 0$ and $e \in GF(q)$.

Proof: Let $P$ be a point of $\mathcal{K}$; if $P = (x, y, z)$ then $f(P) = P \in \mathcal{K}$. Suppose $P = (x, y, 1)$, then $x$ and $y$ are solutions of the following system:

\[
(I) \quad \begin{cases} x^q + y^q + 1 = 0 \\ ax^q + by^q + dx^qy + e + yx^q + fx + fy + fy + fy^q + c = 0 \end{cases}
\]

furthermore $f(P) \in \mathcal{K}$ if and only if $x$ and $y$ are solutions of:

\[
(II) \quad \begin{cases} x^q + y^q + 1 = 0 \\ ax^q + by^q + dx^qy + e + yx^q + fx + fy + fy^q + c = 0 \end{cases}
\]

If we have $d = 0$ and $e \in GF(q)$ the systems $(I)$ and $(II)$ are the same one, means $f(\mathcal{K}) = \mathcal{K}$. Viceversa suppose $f(\mathcal{K}) = \mathcal{K}$, then the systems $(I)$ and $(II)$ must be equivalent and, if $P = (x, y, 1)$ is a point of $\mathcal{K}$, it is necessarily $(x^q - x)(d^qy - dy^q + e^q - c) = 0$. We conclude that $(x, y, 1) \in \mathcal{K}$ implies either $x \in GF(q)$ or $d^qy - dy^q + e^q - c = 0$. Suppose $d \neq 0$, then the polynomial $-dy^q + d^qy + e^q - e \in GF(q^2)[y]$ has at most $q$ solutions in $GF(q^2)$: $y_1, \ldots, y_q$ and the arc $\mathcal{K}$ has at most two points on each line of equation $y = y_i, i = 1, \ldots, q$; observe also that $\mathcal{K}$ has at most two points on each line of equation $x = s, s \in GF(q)$. Therefore we obtain $|\mathcal{K}| \leq 4q$ and this is a contradiction as $q \geq 5$. Then it is necessarily $d = 0$ and $(x^q - x)(e^q - e) = 0$ for every $x$ such that $(x, y, 1) \in \mathcal{K}$. Finally, as $|\mathcal{K}| > 2q$, there exists a point $(x, y, 1) \in \mathcal{K}$ with $x \notin GF(q)$ and this implies $e \in GF(q)$. Denote by $\mathcal{C}$ the class of arcs in $PG(2, q^2)$ obtained by Kestenband’s construction using a matrix $H$ with entries $d = 0$ and $e \in GF(q)$. Then:

**Proposition 3.** The set of points of an arc $\mathcal{K} \in \mathcal{C}$ is the set of points of an arc in the Hall plane of order $q^2$.

Proof: Let $\pi$ be the projective Desarguesian plane $PG(2, q^2)$; fix $t \in GF(q)^* = GF(q) - \{0\}$, let $\mathcal{F}$ be the set of lines with equations $y = mx + kz, m, k \in GF(q^2), m^{q+1} = t$ and let $f$ be the permutation on the point-set of $\pi$ already defined before proposition 2. It is easy to show that $\{\mathcal{F}, f\}$ is a transformation system for $\pi$. Furthermore the transformed plane $\pi^*$ is the Hall plane of order $q^2$ obtained by derivation of $A = AG(2, q^2)$ with derivation set $S_t = \{(1, m, 0) \mid m^{q+1} = t\}$. Finally apply proposition 1 and 2.