ON TWO FUNCTIONAL EQUATIONS RELATED
TO A RESULT OF GRÉGOIRE DE SAINT VINCENT

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ABSTRACT. We give two characterizations of the exponential function by solving two functional equations related to some classical considerations made by Grégoire de Saint-Vincent in the 17th century. Math. Subject Classification: 39B22.
Key words: cube duplication, functional equation, quasiarithmetic means, Jensen equation.

Dedicated to Prof. J. Aczél on the occasion of his 70th birthday.

Grégoire de Saint-Vincent (1647) knew that in order to solve the classical greek problem of duplicating a cube it was necessary to construct the two mean proportionals between two given line segments. For this purpose he used (see (Eves, 1976)) the following result: "the hyperbola drawn through a vertex of a rectangle and having the two sides opposite this vertex for asymptotes meets the circumcircle of the rectangle at a point whose distances from the asymptotes are the mean proportionals between the adjacent sides of the rectangle".

According to figure 1, one sees immediately that using the construction mentioned above one obtains effectively the segments $a^{1/3}b^{2/3}$ and $a^{2/3}b^{1/3}$ which satisfy the relations

$$\frac{a}{a^{2/3}b^{1/3}} = \frac{a^{2/3}b^{1/3}}{a^{1/3}b^{2/3}} = \frac{a^{1/3}b^{2/3}}{b}$$

and consequently in the particular case $a = 2b$ it follows that $a^{1/3}b^{2/3} = 2^{1/3}b$, i.e., the cube of side $a^{1/3}b^{2/3}$ has double volume than that with side $b$.

Thus a natural problem arises. Given $a, b$ in $\mathbb{R}_{++}$, i.e.; $a, b > 0$, consider the point $(a, b)$ and the point $\left(h^{-1}\left(\frac{h(a)+2h(b)}{3}\right), h^{-1}\left(\frac{2h(a)+h(b)}{3}\right)\right)$ involving two conjugated weighted quasi-arithmetic means generated by a bijection $h$ from $\mathbb{R}_{++}$ into $\mathbb{R}$. If both points must
belong to the hyperbola \( y = \frac{ab}{x} \) one obtains the condition:

\[
h^{-1} \left( \frac{h(a) + 2h(b)}{3} \right) \cdot h^{-1} \left( \frac{2h(a) + h(b)}{3} \right) = ab. \tag{1}
\]

On the other hand, if both points belong to the circle of center \( \left( \frac{a}{2}, \frac{b}{2} \right) \) and radius \( \sqrt{a^2 + b^2}/2 \) one obtains

\[
ah^{-1} \left( \frac{h(a) + 2h(b)}{3} \right) + bh^{-1} \left( \frac{2h(a) + h(b)}{3} \right) = h^{-1} \left( \frac{h(a) + 2h(b)}{3} \right)^2 + h^{-1} \left( \frac{2h(a) + h(b)}{3} \right)^2 \tag{2}
\]

The result of de Saint-Vincent implies that if both (1) and (2) hold, necessarily the means involved are the geometric means \( a^{1/3} b^{2/3} \) and \( a^{2/3} b^{1/3} \). Our aim in this paper is to study (1) and (2) separately and see to what extent they characterize these geometric means.

To this end let us introduce \( x = h(a), y = h(b) \) and \( f = h^{-1} : \mathbb{R} \to \mathbb{R}_{++} \), and let us rewrite (1) and (2) in the forms

\[
f \left( \frac{x + 2y}{3} \right) \cdot f \left( \frac{2x + y}{3} \right) = f(x) \cdot f(y), \tag{3}
\]

and

\[
f(x)f \left( \frac{x + 2y}{3} \right) + f(y)f \left( \frac{2x + y}{3} \right) = f \left( \frac{x + 2y}{3} \right)^2 + f \left( \frac{2x + y}{3} \right)^2. \tag{4}
\]

Let us study first equation (3).

**THEOREM 1.** Let \( f \) be a continuous function from \( \mathbb{R} \) into \( \mathbb{R}_{++} \). Then \( f \) satisfies (3) if and only if \( f(x) = Ae^{Bx} \) for some constants \( A, B \) with \( A > 0 \).