A Stability Result for a Magnetostatic Inverse Problem *

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Abstract: For a linearized inverse problem which models a new measurement technique for finding steel reinforcement bars in concrete, we prove a quantitative stability result under physically reasonable a-priori assumptions.

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1 Introduction

This paper is motivated by an industrial cooperation of the first-named author on the inverse problems of finding steel reinforcement bars ("rebars") in concrete walls from measurements of a magnetostatic field scattered by these rebars. In [3], the model is developed and a report about numerical tests is given. It turns out that a certain linear combination of the magnetic field at different points has to be measured in order to "cancel out" the incident field, which is much stronger than the scattered field (see [3], [5]). The model for this measurement process is described in detail in [2]. It is also shown there that a certain linearization of the model leads to a small relative error provided that the total volume of the rebars is small, which is the case in practice.

This linearized model consists of determining a bounded non-negative function \( \mu \) with support contained in a bounded open domain \( \Omega \subset \mathbb{R}^n \) \((n = 2, 3)\) from the following conditions:

Let \( B \) be an open set in \( \mathbb{R}^n \setminus \bar{\Omega} \); for any \( a \in B \), let \( w(\cdot ; a) \) be the solution of

\[
\Delta w(x; a) = \frac{\partial}{\partial x_1} \delta(x - a) \quad \text{in} \, \mathbb{R}^n, \tag{1.1}
\]

\[
w(x; a) \to 0 \quad \text{as} \, |x| \to +\infty \tag{1.2}
\]
and \( v(\cdot; a) \) be the solution of

\[
\Delta v(x; a) = -\text{div}(\mu(x) \text{ grad } w(x; a)) \quad \text{in } \mathbb{R}^n, \tag{1.3}
\]

\[
v(x; a) \to 0 \text{ as } |x| \to +\infty; \tag{1.4}
\]

\( \mu \) has to be determined such that

\[
\frac{\partial^2 v}{\partial x_2 \partial x_1}(a; a) = f(a) \quad \text{for every } a \in B, \tag{1.5}
\]

where \( f \) is a known function, which represents the measurements.

The set \( \Omega \) symbolizes the block of concrete containing the rebars; the magnetic permeability of concrete is normalized to 1, that of steel to \( 1 + \mu \), so that the support of \( \mu \) describes the location of the rebars. The set \( B \) (outside \( \Omega \)) is the set where the permanent magnet which generates the incident field is moved. This (in practice small) magnet is idealized as a point here with unit magnetization, which leads to the right-hand side of (1.1). For a more complete discussion of the model see [2]. In [2], identifiability, i.e., uniqueness of \( \mu \) in the above model, has been proved for the practically relevant three-dimensional case. The problem can be reformulated as an integral equation of the first kind for \( \mu \) and is hence ill-posed. Since the kernel is smooth, this ill-posedness is severe, which has also been demonstrated numerically in [4] (for the two-dimensional case).

In this paper, we investigate the stability of the inverse problem. Due to the ill-posedness, a stability result can only be proven under a-priori assumptions. In the practical problem motivating this work, the rebars are usually circular cylinders whose mutual distances, whose distances to the wall and whose radii are bounded from below by known constants. Since, in addition, one has information about the maximum dimensions of the rebars, these assumptions suffice to bound the number of rebars a-priori. This motivates the a-priori assumptions made below, which will enable us to prove a stability result.

Although we are finally aiming at the three-dimensional case, we consider the two-dimensional case here, where function-theoretic methods are available. Instead of cylinders, we consider circular inclusions. However, since the uniqueness proof of [2] does not apply to the two-dimensional case in an obvious way (see also [4]), we have to give a uniqueness proof for \( n = 2 \) first, which is easy again due to the availability of function-theoretic methods:

Our a-priori assumptions can be formulated such that we assume that \( \mu = \chi_D \), where \( \chi_D \) is the characteristic function of \( D = \bigcup_{j=1}^N D_{P_j,r_j} \), where \( D_{P_j,r_j}, j = 1, \ldots, N \), is a ball with center in \( P_j \) and radius \( r_j \) and \( D_{P_j,r_j} \cap D_{P_k,r_k} = \emptyset \) for \( j \neq k \). In this particular case, a straightforward calculation gives a formula to represent the function \( \frac{\partial^2 v}{\partial x_2 \partial x_1}(a; a) \): We have

\[
\frac{\partial^2 v}{\partial x_2 \partial x_1}(a; a) = \frac{1}{4\pi} \sum_{j=1}^n \frac{r_j^3(a_2 - p_j^2)}{|a - p_j|^3}, \tag{1.6}
\]

where we write \( a = (a_1, a_2) \), \( P_j = (p_j^1, p_j^2) \).

This formula enables us to prove uniqueness by a simple argument: Suppose that \( \mu_1 = \chi_{D_1}, D_1 = \bigcup_{j=1}^N D_{P_j,r_j}, \) and \( \mu_2 = \chi_{D_2}, D_2 = \bigcup_{j=1}^N D_{Q_j,s_j} \), are two different solutions of our inverse