SPECTRA OF PSEUDO-DIFFERENTIAL OPERATORS ON THE
SCHWARTZ SPACE

T.C. CHAU  LIAN PI  M.W. WONG

ABSTRACT. We initiate the study of the spectrum, point spectrum, continuous spectrum and residual spectrum of a constant coefficient pseudo-differential operator on the Schwartz space. The results are in sharp contrast with the corresponding ones for the Banach space $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$. An application to the study of spectra of constant coefficient pseudo-differential operators on the space of Schwartz distributions is given.

1 Introduction

In this paper, we use the standard multi-index notation explained in the books [8] and [14] by Schechter and Wong respectively.

Let $\mathcal{S}$ be the set of all $C^\infty$ complex-valued functions $\varphi$ on $\mathbb{R}^n$ such that for all multi-indices $\alpha$ and $\beta$,

$$\sup_{x \in \mathbb{R}^n} |x^\alpha (D^\beta \varphi)(x)| < \infty.$$ 

It is well-known that $\mathcal{S}$ is a Fréchet space in which the topology is defined by the seminorms $d_{\alpha\beta}$, where $\alpha$ and $\beta$ are multi-indices, and

$$d_{\alpha\beta}(\varphi) = \sup_{x \in \mathbb{R}^n} |x^\alpha (D^\beta \varphi)(x)|, \quad \varphi \in \mathcal{S}.$$ 

Let $m \in (-\infty, \infty)$. We denote by $S^m$ the set of all $C^\infty$ complex-valued functions $\sigma$ on $\mathbb{R}^n$ such that for all multi-indices $\alpha$, there exists a positive constant $C_\alpha$ for which

$$|(D^\alpha \sigma)(\xi)| \leq C_\alpha (1 + |\xi|)^{m-|\alpha|}, \quad \xi \in \mathbb{R}^n.$$ 

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We call any $\sigma$ in $\bigcup_{m \in \mathbb{R}} S^m$ a symbol. Let $\sigma$ be a symbol. Then we define the linear operator $T_\sigma$ on the Schwartz space $S$ by
\[
(T_\sigma \varphi)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \xi} \sigma(\xi) \hat{\varphi}(\xi) d\xi, \quad x \in \mathbb{R}^n
\]
for every function $\varphi$ in $S$, where
\[
\hat{\varphi}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix \xi} \varphi(x) dx, \quad \xi \in \mathbb{R}^n.
\]
We call $T_\sigma$ the pseudo-differential operator corresponding to the symbol $\sigma$. We note that $T_\sigma : S \to S$ is a continuous linear operator. A discussion of pseudo-differential operators on the Schwartz space can be found in the book [14] by Wong.

We define the resolvent set $\rho(T_\sigma)$ of the pseudo-differential operator $T_\sigma : S \to S$ on the Schwartz space $S$ to be the set of all complex numbers $\lambda$ for which the operator $T_\sigma - \lambda I : S \to S$ is one to one and onto, where $I : S \to S$ is the identity operator. The spectrum $\Sigma(T_\sigma)$ of the operator $T_\sigma : S \to S$ is defined to be the complement of $\rho(T_\sigma)$ in the set $\mathbb{C}$ of all complex numbers.

The impetus for initiating a spectral theory of pseudo-differential operators on the Schwartz space stems from the fact that we want to have a structural theory of pseudo-differential operators as continuous linear operators on $S$ rather than discontinuous (unbounded) linear operators on $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$. The spectral theory of partial and pseudo-differential operators on $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, has been studied by Schechter [8], Talenti [9], Thompson [10] and Wong [11,12,13,14].

Following the book [2] by Dowson or the book [3] by Dunford and Schwartz on spectral theory of linear operators, we introduce the following notions for a pseudo-differential operator $T_\sigma : S \to S$ on the Schwartz space $S$.

The point spectrum $\Sigma_p(T_\sigma)$ of the operator $T_\sigma : S \to S$ is defined to be the set of all eigenvalues of the operator, i.e.,
\[
\Sigma_p(T_\sigma) = \{ \lambda \in \mathbb{C} : T_\sigma - \lambda I \text{ is not one to one} \}.
\]

The continuous spectrum $\Sigma_c(T_\sigma)$ of the operator $T_\sigma : S \to S$ is defined to be the set of all complex numbers $\lambda$ such that the operator $T_\sigma - \lambda I : S \to S$ is one to one, the range $R(T_\sigma - \lambda I)$ of $T_\sigma - \lambda I : S \to S$ is dense in $S$, but not equal to $S$.

The residual spectrum $\Sigma_r(T_\sigma)$ of the operator $T_\sigma : S \to S$ is defined to be the set of all complex numbers $\lambda$ such that the operator $T_\sigma - \lambda I : S \to S$ is one to one and the range $R(T_\sigma - \lambda I)$ of $T_\sigma - \lambda I : S \to S$ is not dense in $S$.

It is obvious from the definitions that the point spectrum, continuous spectrum and residual spectrum of the pseudo-differential operator $T_\sigma : S \to S$ are mutually disjoint and their union is equal to the spectrum of $T_\sigma : S \to S$.

Let us end the introduction by saying a few words on the genesis of the paper. In Section 2, we determine the spectrum of a pseudo-differential operator $T_\sigma : S \to S$ in terms of its symbol $\sigma(\xi)$ for a class of pseudo-differential operators which includes all the strongly Carleman operators studied in the two papers [12,13] by Wong. In Section 3, we study the point spectrum of a pseudo-differential operator on the Schwartz space by making use of a