P-ADIC SEMI-MONTEL SPACES AND POLAR INDUCTIVE LIMITS

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ABSTRACT. In this paper we give a characterization of p-adic semi-Montel spaces which allows us to describe the finest polar semi-Montel topology coarser than the original topology of a locally convex space. As an application we derive that every polar semi-Montel space is a polar inductive limit of a family of nuclear spaces. We also pay attention to the connection with compactifying operators.


§1. INTRODUCTION.

1. Throughout this paper \( \mathbb{K} \) is a complete non-archimedean non-trivially valued field with valuation \( |.| \). For the basic notions concerning locally convex spaces over \( \mathbb{K} \), not explicity mentioned below, we refer to [7].

WE ONLY CONSIDER LOCALLY CONVEX SPACES \( E \) OVER \( \mathbb{K} \) WHICH ARE HAUSDORFF AND SUCH THAT THEIR TOPOLOGICAL DUAL \( E' \) SEPARATES THE POINTS OF \( E \).

2. Let \( E \) be a locally convex space. A subset \( X \) of \( E \) is called compactoid if for every zero-neighbourhood \( U \) in \( E \)

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there exists a finite set $S$ in $E$ such that $X \subset \text{co}S + U$, where $\text{co}S$ is the absolutely convex hull of $S$. Obviously every compactoid set is bounded. If in $E$ every bounded subset is compactoid then $E$ is called semi-Montel (SM) space (some equivalent definitions are given in Theorem 2.1).

If $E$ is a normed space then $E$ is SM iff $E$ is finite-dimensional. The best known class of infinite-dimensional SM-spaces consists of the nuclear spaces (definition below). In this paper we study further relations between SM-spaces and nuclearity by means of inductive limits. In fact we put ourselves in a more general position. Starting with a locally convex space $E, \tau$ we construct the finest SM-topology, $\tau_{sm}$, on $E$ coarser than $\tau$ and tackle our problem through the SM-space $E, \tau_{sm}$. At certain points we assume our spaces to be polar (i.e. that they have a fundamental system of zero-neighbourhoods consisting of absolutely convex sets $U$ with $U = U_{oo}$, $U_{oo}$ being the bipolar of $U$). The fundamental reason behind this is that every nuclear space is polar whereas an SM-space $E$ needs not to be polar even when its dual $E'$ separates the points of $E$, as we show in the following example.

**Example.** First let $E$ be any infinite-dimensional $K$-vector space and consider on $E$ the finest locally convex topology $\tau(E,E^*)$. A basis of zero-neighbourhoods for this topology consists of all the absolutely convex and absorbing subsets of $E$. It is easy to see that $(E,\tau(E,E^*))' = E^*$, the algebraic dual of $E$, and that every $\tau(E,E^*)$-bounded subset of $E$ is finite-dimensional. Hence, $(E,\tau(E,E^*))$ is a polar SM-space.

Take now $E=l^\infty$ and suppose $K$ is not spherically complete. Let $P:E \to E/\text{c}_0$ be the canonical surjection and define a seminorm $q$ on $E$ by $q(x) = \|P(x)\|_{E/\text{c}_0}$ $(x \in E)$. Finally, consider on $E$ the Hausdorff topology $\tau$ generated by all the $\sigma(E,E^*)$-continuous seminorms and $q$ (where $\sigma(E,E^*)$ denotes the weak topology corresponding to $\tau(E,E^*)$). Then obviously $\sigma(E,E^*) \subset \tau \subset \tau(E,E^*)$. Hence $(E,\tau)'$ separates the points of $E$ and by [7], 7.5 we have that $E,\tau$ is an SM-space. However the space $E,\tau$ is not a polar space (see [4], p.6).

3. A continuous linear map $(T \in \mathcal{L}(E,F))$ from $E$ to a locally convex space $F$ is said to be compactifying $(T \in \mathcal{C}(E,F))$