A NEW CHARACTERIZATION OF GUNDERSEN'S EXAMPLE OF TWO MEROMORPHIC FUNCTIONS SHARING FOUR VALUES

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ABSTRACT. Let \( f \) and \( g \) be nonconstant meromorphic functions sharing four values \( 1M \) and satisfying \( f^{-1}(\{a\}) \subset g^{-1}(\{b\}) \) for two values \( a, b \) not shared by \( f \) and \( g \). Then either 
\[ f = T \circ g \]  
with a Möbius transformation \( T \) or 
\[ f = L \circ \hat{f} \circ h \]  
and 
\[ g = L \circ \hat{g} \circ h, \]  
where 
\[ \hat{f}(z) = (\exp z + 1)/(\exp z - 1)^2 \]  
and 
\[ \hat{g}(z) = (\exp z + 1)^2/(8(\exp z - 1)) \]  
are the functions in Gundersen's example [1], \( L \) is a Möbius transformation and \( h \) is an entire function.

1. INTRODUCTION AND RESULTS

In this paper the term "meromorphic" will always mean meromorphic in the complex plane \( \mathbb{C} \). Two meromorphic functions \( f \) and \( g \) share the value \( a \in \mathbb{C} \) IM (ignoring multiplicities), if \( f^{-1}(\{a\}) = g^{-1}(\{a\}) \). If, in addition, every \( p \)-fold \( a \)-point \( z_0 \) of \( f \) is also a \( p \)-fold \( a \)-point of \( g \), \( p = p(z_0) \), then \( f \) and \( g \) share the value \( a \) CM (counting multiplicities).

We will use the standard notations and results of the Nevanlinna theory (see [3] for example).

R. Nevanlinna proved the following two classical uniqueness theorems:

**Theorem A (Five-point-theorem [5, 6])**. If \( f \) and \( g \) are nonconstant meromorphic functions sharing five distinct values \( 1M \) then \( f = g \).

**Theorem B (Four-point-theorem [5, 6])**. If \( f \) and \( g \) are distinct nonconstant meromorphic functions sharing four distinct values \( a_1, \ldots, a_4 \) CM, then \( f = T \circ g \) with a Möbius transformation \( T \). Moreover, two of the four values, \( a_3 \) and \( a_4 \), say, are Picard exceptional values of \( f \) and \( g \), and the cross-ratio \( (a_1, a_2, a_3, a_4) \) equals \(-1\).

Theorem B was improved by Gundersen:

**Theorem C ([2])**. If \( f \) and \( g \) are distinct nonconstant meromorphic functions sharing two values \( CM \) and another two values \( 1M \), then the conclusion of Theorem B remains valid.

However, Gundersen showed that the condition CM cannot be dropped completely in Theorem B. In [1] he gave the following example: let

\[
\hat{f}(z) := \frac{e^z + 1}{(e^z - 1)^2} \quad \text{and} \quad \hat{g}(z) := \frac{(e^z + 1)^2}{8(e^z - 1)}. \tag{1}
\]

Then \( \hat{f} \) and \( \hat{g} \) share the values 0, 1, \( \infty \) and \(-1/8\). All zeros and 1-points are simple for \( \hat{f} \) and double for \( \hat{g} \), whereas all poles and \(-1/8\)-points are double for \( \hat{f} \) and simple for \( \hat{g} \). In particular, none of the values is shared CM and the conclusion of Theorem B does not hold.

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For each shared value \( a \in \{0, 1, \infty, -1/8\} \), either \( \hat{f} \) or \( \hat{g} \) has only multiple \( a \)-points. Conversely, \( \hat{f} \) and \( \hat{g} \) is (up to a Möbius transformation and composition with an entire function) the only pair of functions with this property:

**Theorem D ([7]).** Let \( f \) and \( g \) be nonconstant meromorphic functions sharing four distinct values \( a_1, \ldots, a_4 \). Assume that for every \( \nu = 1, \ldots, 4 \) either \( f \) or \( g \) has only multiple \( a_\nu \)-points. Then

\[
f = L \circ \hat{f} \circ h \quad \text{and} \quad g = L \circ \hat{g} \circ h
\]

with a Möbius transformation \( L \) and an entire function \( h \).

The functions \( \hat{f} \) and \( \hat{g} \) in Gundersen’s example have another interesting property. Since

\[
\hat{f}(z) + \frac{1}{2} = \frac{e^{2z} + 3}{2(e^z - 1)^2} \quad \text{and} \quad \hat{g}(z) - \frac{1}{4} = \frac{e^{2z} + 3}{8(e^z - 1)}
\]

we have

\[
\hat{f}(z) = -\frac{1}{2} \iff \hat{g}(z) = \frac{1}{4}.
\]

In this paper we determine all pairs of functions \( f \) and \( g \) sharing four values and satisfying

(2)

\[
f(z) = a \implies g(z) = b
\]

for two values \( a \) and \( b \) not shared by \( f \) and \( g \). Of course, if \( f = T \circ g \) with a Möbius transformation \( T \) then (2) holds whenever \( a = T(b) \) and \( b \) is an arbitrary complex value. Apart from this trivial case, property (2) characterizes Gundersen’s example uniquely (up to a Möbius transformation and composition with an entire function):

**Theorem 1.** Let \( f \) and \( g \) be distinct meromorphic functions sharing four distinct values \( a_1, \ldots, a_4 \) IM. If there exist \( a, b \in \hat{C} \setminus \{a_1, \ldots, a_4\} \) satisfying (2), then either

(3)

\[
f = T \circ g
\]

with a Möbius transformation \( T \) or

(4)

\[
f = L \circ \hat{f} \circ h \quad \text{and} \quad g = L \circ \hat{g} \circ h
\]

with a Möbius transformation \( L \) and an entire function \( h \).

2. PRELIMINARIES FOR THE PROOF OF THEOREM 1

Let \( f \) and \( g \) be distinct nonconstant meromorphic functions sharing four distinct finite values \( a_1, \ldots, a_4 \). For \( r > 0 \) let \( T(r) := \max\{T(r, f), T(r, g)\} \). We write \( \phi(r) = S(r) \) for every function \( \phi : (0, \infty) \to \mathbb{R} \) satisfying \( \phi(r)/T(r) \to 0 \) for \( r \to \infty \) possibly outside a set of finite Lebesgue measure.

We call \( z_0 \) a \((p, q)\)-fold \( a_\nu \)-point (of \( f \) and \( g \)) if \( f(z_0) = a_\nu \) with multiplicity \( p \) and \( g(z_0) = a_\nu \) with multiplicity \( q \). \( N_{(p, q)}(r, a_\nu) \) denotes the counting function of all the \((p, q)\)-fold \( a_\nu \)-points where each point is counted once.

Following [4], we define

\[
\psi = \frac{f'g'(f-g)^2}{(f-a_1)(f-a_2)(f-a_3)(f-a_4)(g-a_1)(g-a_2)(g-a_3)(g-a_4)}.
\]