ON CONFORMAL ROTATIONS OF SIMPLY-CONNECTED DOMAINS

Gerald Schmieder

Introduction

In a recent paper about conformal automorphisms of simply-connected bounded domains in the complex plane, D. Gaier [1] raised the following question:

Let \( f \) be a holomorphic univalent and bounded function in the unit disk \( \mathbb{D} \) and \( \alpha_n \neq 0 \) be a sequence of real numbers tending to 0 such that the functions \( f(e^{i\alpha_n}z) \) converge to \( f(z) \) uniformly on \( \mathbb{D} \); does this imply \( f(e^{i\alpha}z) \to f(z) \) uniformly on \( \mathbb{D} \) for \( \alpha \to 0 \) ?

Gaier [1] gives an affirmative answer in the case that the image domain \( f(\mathbb{D}) \) has only prime ends of the first kind or that there is a prime end \( P \) not of the first kind which has a neighbourhood (in the usual prime end topology, cf. [3]) of prime ends only of the first kind except \( P \) itself; the latter case is trivial, since there is no such sequence \( \alpha_n \) with the property described above.

In this paper it is shown that the answer is negative in the general case. An example is constructed by induction based on an idea due to Herzog and Piranian [2].

1. Spike functions of first type

For \( \beta \in (0, \pi) \), \( \rho \in (0,1) \) let \( S(\beta) = \{ z \in \mathbb{D} \mid -\beta < \arg z < \beta \} \) and \( K(\beta, \rho) = S(\beta) - \{ z \in \mathbb{D} \mid |z| < \rho \} \). Take some \( \delta > 0 \) and \( \beta, \rho \) fixed. For some \( \lambda > 0 \) consider the function \( g(z) = 1 - (1-\zeta)^\lambda \) in \( \mathbb{D} \). For \( \lambda \) sufficiently small we obtain:

a) \( |g| < \delta \) in \( \mathbb{D} - K(\beta, \rho) \), b) \( \Re g' > 0 \) in \( \mathbb{D} \), c) \( g(0) = 0 \), \( g(1) = 1 \).

Let \( m \in \mathbb{N} \) be so large that \( \alpha := \frac{2\pi}{m} > 2\beta \). The function

\[
G(z) = \sum_{\nu=0}^{m-1} e^{i\nu\alpha} g(ze^{-i\nu\alpha})
\]

is holomorphic in \( \mathbb{D} \) and continuous in \( \overline{\mathbb{D}} \). By a well known argument it follows from b) that \( G \) is univalent in \( \mathbb{D} \). Because all the segments obtained from \( K(\beta, \rho) \) under rotations by integer multiples of \( \alpha \) are pairwise disjoint or identical, we conclude
(I) \( |G| \leq 1 + (m-1)\delta \) in \( \overline{\mathbb{D}} \).

Direct computation gives \( (z \in \mathbb{D}) \)

(II) \( G(z e^{i\alpha}) = e^{i\alpha} G(z) \).

Since \([0,1) \subset g(\mathbb{D})\) there is some \( z_0 \in [0,1) \) with \( g(z_0) = 1 - \delta \) (assuming \( \delta < 1 \)) and a) gives the estimate

(III) \( |G(z_0 e^{i\alpha/2}) - G(z_0)| \geq |G(z_0)| - |G(z_0 e^{i\alpha/2})| > 1 - \delta - (m-1)\delta - m\delta = 1 - 2m\delta \).

Figure 1 shows the image domain \( G(\mathbb{D}) \).

2. Spike functions of second type

In this paragraph we will construct an unbounded function \( H \) similar to \( G \) but having infinitely many "spikes" and such that there is a sequence of rotations of \( \mathbb{D} \) tending to \( \text{id} \) with the property that the effect of these rotations composed with \( H \) can prescribed arbitrarily small whereas for another sequence of rotations \( (\sigma_n \to 0) \) \( H(z e^{i\sigma_n}) \) does not converge to \( H(z) \) uniformly on \( \mathbb{D} \).

Let a sequence \( \varepsilon_j > 0 \) be given.

By induction we will prove the existence of functions \( G_j \) with

(i) \( G_j : \mathbb{D} \to \mathbb{C} \) is univalent and continuous on \( \overline{\mathbb{D}} \),

(ii) \( |G_j(z)| < j + \frac{1}{2} \) \( (z \in \mathbb{D}) \),