ON THE EPSTEIN UNIVALENCE CRITERION

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C.L. Epstein has recently found a remarkable new univalence criterion [2][3] which includes most previously known criteria. This is a byproduct of his work on surfaces in hyperbolic three-space and his proof is quite different from classical approaches which use quasiconformal mappings or differential equations.

We shall give a shorter proof of the analytic function case of Epstein's theorem by the Löwner differential equation.

I wish to thank Professor Charles Epstein for our discussion.

Let \( \mathbb{D} \) be the unit disk in \( \mathbb{C} \) and let

\[
S_f(z) = \frac{d}{dz} \left( \frac{f''(z)}{f'(z)} - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2 \right)
\]
denote the Schwarzian derivative.

**THEOREM (C.L. Epstein).** Let \( f \) be meromorphic and \( g \) analytic in \( \mathbb{D} \). If both functions are locally univalent in \( \mathbb{D} \) and if

\[
(1) \quad \frac{1}{2} |1 - |z|^2| \cdot [S_f(z) - S_g(z)] + (1 - |z|^2) \cdot \frac{z g''(z)}{g'(z)} \leq 1
\]

for \( z \in \mathbb{D} \) then \( f \) is univalent in \( \mathbb{D} \).

Epstein has proved a more general result [2,Theorem 7.2] [3,Theorem A] and the present theorem [3,formula (2.2)] is the special case that his real-valued function is harmonic. He made however the very mild additional assumption [3,formula (2.1)] that

\[
(2) \quad (1 - |z|^2) \cdot \left| z \frac{g''(z)}{g'(z)} \right| \leq \max(1,2|z|^2) \quad (z \in \mathbb{D})
\]

which we will not need in our proof.

If \( g(z) = z \) then condition (1) becomes

\[
(1 - |z|^2)^2 |S_f(z)| \leq 2; \text{ this is Nehari's univalence criterion [4].}
\]
If \( f = g \) then (1) becomes \((1-|z|^2)|zf''(z)/f'(z)| \leq 1\); this is Becker's univalence criterion [1], and (2) is automatically satisfied in this case.

**Proof.** If \( f(z) = a_0 + a_1 z + \ldots, g(z) = b_0 + b_1 z + \ldots \) then \( a_1 \neq 0, b_1 \neq 0 \) and we may consider

\[
f^* = \frac{f - a_0}{a_1 + (a_2/a_1-b_2/b_1)(f-a_0)}, \quad g^* = \frac{g - b_0}{b_1}
\]

instead of \( f \) and \( g \). These normalizations do not affect our assumptions and imply that \( f(z) = g(z) + O(z^3) \) and therefore \( f'(z)/g'(z) = 1 + O(z^2) \) as \( z \to 0 \).

We introduce now

\[
(3) \quad v(z) = \sqrt{g'(z)/f'(z)} = 1 + \beta z^2 + O(z^3),
\]

\[
(4) \quad u(z) = f(z)v(z) = z + \alpha z^2 + O(z^3).
\]

Both functions are analytic in \( \mathbb{D} \) because \( f \) cannot have multiple poles and because \( f' \) and \( g' \) cannot have zeros. For \( 0 \leq t < \infty \) we consider [1,p.38]

\[
(5) \quad f_t(z) = \frac{u(e^{-t}z) + (e^t-e^{-t}) zu'(e^{-t}z)}{v(e^{-t}z) + (e^t-e^{-t}) zv'(e^{-t}z)}.
\]

This function is meromorphic in \( z \in \mathbb{D} \). By (3), the denominator is \( 1 + O(z) \) as \( z \to 0 \) uniformly in \( t \). Hence there are constants \( r_0 > 0 \) and \( K_0 \) such that

\[
(6) \quad |f_t(z)| \leq K_0e^t \quad \text{for} \quad |z| < r_0, \quad t \geq 0.
\]

By (4), the numerator in (5) is \( e^t z + O(z^2) \) as \( z \to 0 \) and we conclude that

\[
(7) \quad f_t(z) = e^t z + O(z^2) \quad \text{as} \quad z \to 0.
\]

We write \( f'_t = \partial f_t/\partial z \) and \( f_t = \partial f_t/\partial t \) and obtain from (5) by computation that