Normal families of gap power series

St. Ruscheweyh and K.-J. Wirths

The following heuristic principle is attributed to A. Bloch and has been explicitly stated in Hille's book [3, p.250] (for the more general meromorphic case): A family of holomorphic functions which have a property \( P \) in common in a domain \( D \) is (apt to be) a normal family in \( D \) if \( P \) cannot be possessed by a nonconstant entire function in the finite plane. Of course, since the term 'property' is not well-defined, this principle is not a conjecture, let alone a theorem. Nevertheless, many examples for normal families can be viewed as special cases of this principle. Zalcman [7] proved the principle for a set of 'properties' covering some of the better known cases. A basic ingredient in his proof is an invariance under linear transformations of the families under consideration. In the present note we formulate - as a conjecture - a case of Bloch's principle which is not covered by Zalcman's theorem and we verify the conjecture in a number of cases.

Let \( T \) be a finite or infinite system of integers \( 0 = m_0 < m_1 < \ldots \) and let \( A_T \) denote the set of analytic functions in the unit disc \( \mathbb{D} \) having a power series expansion

\[
f(z) = \sum_{k \in T} a_k z^k.
\]

Furthermore, let \( A_T^0 \) consist of those functions \( f \in A_T \) which satisfy \( f(z) \neq 0 \) in \( \mathbb{D} \).

**Conjecture:** \( A_T^0 \) is a normal family in \( \mathbb{D} \) if and only if there is no non-constant entire function \( f \in A_T \) which is non-vanishing in \( \mathbb{C} \).
This case of Hille's principle is also closely related to a conjecture discussed in [4] where an application to the convolution duality of analytic functions is the main subject (compare also [6]). The "only if" part of the conjecture is trivial. In fact, assume such a function \( \tilde{f} \) exists and let \( f_n(z) = \tilde{f}(nz) \) for \( n \in \mathbb{N} \). Then \( \{ f_n \} \subseteq \mathcal{A}_T^O \) and obviously cannot contain a subsequence which converges locally uniformly to a limit function or to \( \infty \).

The simplest cases where no \( \tilde{f} \) exist are those with \( T \) finite so that \( \mathcal{A}_T^O \) contains polynomials only. Then the normality of \( \mathcal{A}_T^O \) follows immediately from the following well-known lemma which is easily verified by induction:

**Lemma 1:** Let \( P(z) = \sum_{k=0}^{n} a_k z^k \neq 0 \) in \( \mathbb{D} \). Then

\[
|a_k| \leq \binom{n}{k} |a_0|, \quad k = 1, \ldots, n. \tag{2}
\]

The main result of this note is to establish the conjecture for the cases of Fejér gaps, i.e. for those infinite \( T \) with

\[
\sum_{k=1}^{\infty} \frac{1}{m_k} < \infty. \tag{3}
\]

**Theorem:** Let \( T \) satisfy (3). Then \( \mathcal{A}_T^O \) is normal in \( \mathbb{D} \).

Our proof rests on the following lemma due to Fekete [1], compare Pommerenke [5, p. 152].

**Lemma 2:** Let \( T \) satisfy (3) and

\[
f(z) = \sum_{k=0}^{\infty} a_{m_k} z^{m_k} \in \mathcal{A}_T^O. \tag{4}
\]

Then for \( j \in \mathbb{N} \) we have

\[
\sum_{k=0}^{j} \sum_{k=0}^{m_k} \prod_{k=0}^{j} a_{m_k} \in \mathcal{A}_T^O
\]

where