Two results on arithmetical functions

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Abstract: Generalizing two results of Rieger [8] and Selberg [10] we give asymptotic formulas for sums of type

\[ \sum_{n \leq x, n \equiv a \pmod{k}} \chi(n) \] and \[ \sum_{n \leq x, n \equiv a \pmod{k}} \chi(n), \]

where \( \chi \) is a suitable multiplicative function, \( f_1, \ldots, f_r \) are “small” additive, prime-independent arithmetical functions and \( k, l \) are coprime. The proofs are based on an analytic method which consists of considering the Dirichlet series generated by \( \chi(n)z_1^{f_1(n)} \cdots z_r^{f_r(n)}, z_1, \ldots, z_r \) complex.

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1 Introduction

Selberg [10] proved in 1954

\[ \sum_{n \leq x, \omega(n) = k} 1 = x Q(\log \log x) / \log x + O \left( x (\log \log x)^k (\log x)^{-2} \right), \]

where \( Q(x) \) is a polynomial of degree \( \leq k - 1 \).

Rieger [8] used Selberg’s method to prove

\[ \sum_{n \leq x, \omega(n) = qr} 1 = x/(kq) + O \left( x (\log x)^{\text{Re}(\exp(2\pi i/q) - 1)} \right) + O \left( \frac{x}{\log x} \right) \]

for natural numbers \( k, l, q, r \), where \( k, l \) are coprime.

To estimate sums of type

\[ (1) \sum_{n \leq x, n \equiv a \pmod{k}} \chi(n) \quad \text{or} \quad \sum_{n \leq x, n \equiv a \pmod{k}} \chi(n), \]

\( f_1, \ldots, f_r \) are suitable prime-independent additive functions and \( k, l \) are coprime.
where $\chi$ is a suitable multiplicative function, $f_1, \ldots, f_r$ are additive functions, taking only nonnegative integer values and $k, l$ coprime, we first have to give an approximation of

$$F_{k, l}(x; z_1, \ldots, z_r) := \sum_{n \leq x \atop n \equiv i \pmod{k}} \chi(n) z_1^{f_1(n)} \cdots z_r^{f_r(n)}.$$  

Choosing $z_1, \ldots, z_r$ appropriately in (2) and using an orthogonality relation we derive an estimation for the first sum in (1).

Since $f_1, \ldots, f_r$ take only nonnegative integer values, $F_{k, l}(x; z_1, \ldots, z_r)$ is a polynomial in $z_1, \ldots, z_r$, where the coefficient of $z_1^{z_1} \cdots z_r^{z_r}$ is the second sum in (1).

2 Preliminaries

Definition We call a $(r + 1)$-tupel $(\chi, f_1, \ldots, f_r)$ of arithmetical functions admissible, if

(i) $\chi$ is multiplicative, such that there is an $\alpha \in \mathbb{R}_+$ with $\chi(p) = \alpha$ for all prime $p$.

(ii) $f_1, \ldots, f_r$ are additive, taking only nonnegative integers and $f_i(p) = 1$ for $i = 1, \ldots, r$ and every prime $p$.

(iii) For $\varrho \in \mathbb{R}_{\geq 0}$ we define

$$\sigma_0(\varrho) := \inf_{\varrho > 1/2} \left\{ \sum_p \sum_{\nu \geq 2} \left| \chi(p^\nu) \right| \varrho^{f_1(p^\nu) + \cdots + f_r(p^\nu)} p^{-\nu \varrho} < \infty \right\},$$

where we demand $1 < R := \sup \{ \varrho \geq 0 \mid \sigma_0(\varrho) < 1 \}$.

Furthermore we define $z := (z_1, \ldots, z_r) \in \mathbb{C}^r$ and $z^u := z_1^{u_1} \cdots z_r^{u_r}$. Throughout this article we write $\mathbb{N}$ for the positive and $\mathbb{N}_0$ for the nonnegative integers.

Remark

- The condition (ii) can be weakened only to assume $f_i(p) = f_i(2) \neq 0$ and $f_i(n)/f_i(2)$ nonnegative integral, where $f_i(p)$ is not necessarily an integer. (cf. [4]).

- The condition $R > 1$ is necessary, since we have to analyze some in $|z| < R$ analytic function on $|z| = 1$ (cf. Delange [2] for the definiton of $R$).

A sufficient condition for $R > 1$ is, if for an $\varepsilon \in [0, 1/2]$ and every $k \in \mathbb{N}_0$ (cf. Theorem 5.3 in [1])

$$\chi(n) \ll n^\varepsilon \quad \text{and} \quad f_i(p^k) \ll k$$

holds. Such functions are called prime–independent.

The triple $(\chi, \omega, \Omega)$ with $\chi \in \{1, \mu^2, \tau_\alpha \}$ is therefore admissible, where $\mu$ is the Möbius function, $\tau_\alpha(n)$ is the extension of the divisor function $\tau$ (cf. [1], [11] for the definition of $\tau_\alpha$). By $\omega(n)$ (resp. $\Omega(n)$) we denote the number of different (with multiplicity) prime factors of $n$. Other interesting examples of admissible functions one can find in [3], [11] or [9].