Stability of the Cauchy type equations
in $\mathcal{L}_p$ norms

Józef Tabor

Abstract

Let $(X, +, \mu)$ be a measurable group such that $\mu$ is complete and $\mu(X) = \infty$, and let $(E, +)$ be a metric group. Let $f : X \to E$ be any mapping. We prove that if there exists a $p > 0$ such that the function $(d(f(x + y), f(x) + f(y)))^p$ is majorizable by an integrable function then $f$ is almost everywhere additive. Similar results we also obtain for the Jensen and Pexider equations.

1 Introduction

In 1940 S. M. Ulam posed the following problem (cf. [7]). We are given a group $(X, +)$ and a metric group $(Y, +, d)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if $f : X \to Y$ satisfies

$$d(f(x + y), f(x) + f(y)) < \delta \quad \text{for all } x, y \in X$$

then a homomorphism $a : X \to Y$ exists with

$$d(f(x), a(x)) < \varepsilon \quad \text{for all } x \in X?$$

This question became a source of the stability theory in the Hyers-Ulam sense. The most important results and the large list of references concerning this theory are included in the survey papers [2], [3], [5].

Assume that $Y$ is a normed space and let $\| \|$ denote the supremum norm in the space of bounded functions defined on $X$ ($X \times X$ respectively) and taking its values in $Y$. Let $Cf$ denote the Cauchy difference for a function $f : X \to Y$, i.e. let

$$Cf(x, y) := f(x + y) - f(x) - f(y) \quad \text{for } x, y \in X.$$ 

Then the stability question can be reformulated as follows. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if $f : X \to Y$ satisfies

$$\|Cf\|_S < \delta$$

1991 Mathematics Subject Classification: Primary 39B72.
then an additive function \( a : X \to Y \) exists with

\[
||f - a||_S < \varepsilon?
\]

There is no reason to consider such a problem only for the supremum norm. Other norms in function spaces are used very often as well in mathematics as in its applications. Therefore it is very natural to consider Ulam's question for different norms. In particular the \( L_p \) norm seems to be very interesting.

In this paper we study the stability of the Cauchy, Jensen and Pexider equations in a certain generalization of \( L_p \) norm.

2 Preliminaries

In this section we establish some denotations and the general assumptions. We also prove a few introductory lemmas. Throughout the paper we assume that \((X, +, \Sigma, \mu)\) is a complete measurable group, i.e. that

(a) \((X, +)\) is a group,

(b) \((X, \Sigma, \mu)\) is a \(\sigma\)-finite measure space, \(\mu\) is not identically zero and complete,

(c) the \(\sigma\)-algebra \(\Sigma\) and the measure \(\mu\) are invariant with respect to left translations,

(d) \(\mu \times \mu\) is the completion of the product measure,

(e) the transformation

\[
S : X \times X \ni (x, y) \mapsto (x, x + y)
\]

is measurability preserving, i.e. \(S\) and \(S^{-1}\) are measurable.

Additionally, we assume that \(\mu(X) = \infty\).

The space \(\mathbb{R}^n\) with the Lebesgue measure is an example of a group satisfying the above conditions.

It is worth mentioning that under the assumptions (a) - (e) the measure \(\mu\) is invariant under translations and under symmetry with respect to zero, and that the transformations \(S\) and \(S^{-1}\) preserve the measure \(\mu \times \mu\) (cf. [4], §59). We will be applying these facts very often.

If mappings \(f\) and \(g\) are equal almost everywhere with respect to the measure \(\mu\) \((\mu \times \mu)\) we write \(f \overset{\mu}{=} g\) \((f \overset{\mu \times \mu}{=} g\), respectively).

Let \((Y, \nu)\) be a measure space. By \(L_1(Y, \mathbb{R})\) we denote the set of all integrable functions defined on \(Y\) and taking their values in \(\mathbb{R}\). For non-negative function \(f : Y \to \mathbb{R}\) we define its "upper integral". We put

\[
\int_Y^+ f d\nu := \inf \left\{ \int_Y \varphi d\nu : \varphi \in L_1(Y, \mathbb{R}), f(x) \leq \varphi(x) \right\}.
\]

If there is no integrable majorant for \(f\) then we write

\[
\int_Y^+ f d\nu = \infty.
\]

The following properties of the upper integral result directly from its definition (by \(0 \cdot \infty\) we mean \(0\)):

1° \(c \geq 0, f(x) \geq 0 \Rightarrow \int_Y^+ c f(x) d\nu = c \int_Y^+ f(x) d\nu,\)