A GEOMETRIC VIEW ON RADON TRANSFORMS

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1 Introduction

Radon transforms go back to P. Funk (1916) and J. Radon (1917). Further developments are due to F. John (1934). Nowadays evolution is mainly due to S. Helgason, I.M. Gelfand and their schools. For historical remarks, methods, results and references on Radon transforms we refer to S. Helgason [9], resp. [7].

The aim of this note is to present a new geometric view on inversion formulas for Radon transforms in spaces of constant curvature. Let $X^n_\kappa$ be the n-dimensional real standard space of constant curvature $\kappa$, i.e. either the euclidean space $E^n$ ($\kappa = 0$), or the sphere $S^n_\kappa$ ($\kappa > 0$), or the hyperbolic space $H^n_\kappa$ ($\kappa < 0$). Let $G^{k,n}$ be the homogeneous space of all nonoriented $k$-dimensional totally geodesic subspaces of $X^n_\kappa$ (for short called $k$–planes) $(0 \leq k \leq n)$.

The $k$-plane Radon transform $R^k$ maps $C^\infty_c(X^n_\kappa)$ into $C^\infty_c(G^{k,n})$, namely

$$ (R^k f)(\xi) = \int_{G^{k,n}} f(x) \, dx \ , \ \xi \in G^{k,n} \; ; $$

($C^\infty_c$ denotes the space of $C^\infty$-functions with compact support; $dx$ the riemannian volume density on the $k$-plane $\xi$).

Section 2 is about the euclidean case. We start at the geometric point (3), and using integralgeometric techniques we obtain a reduction formula for euclidean Radon transforms (2)(proposition 2.1), i.e. the reduction step from $R^k f$ to $R^{k-2} f$. By successive application of the reduction step we get from this the classical inversion formulas. The inversion formulas
(6)(proposition 2.2), resp. (7)(proposition 2.3) contain the basic cases \( k = 1, n = 3 \), resp. \( k = 1, n = 2 \).

Section 3 is about the non-euclidean case. We obtain the 2-step reduction formula (10)(proposition 3.1), and an inversion formula (18)(proposition 3.2) in case \( k = 1, n = 3 \).

2 The euclidean case \( \kappa = 0 \)

Consider the euclidean space \( E^n \). Taking into account euclidean parallel translations we view \( G^{k,n} \) fibered over the base space \( G_0^{k,n} \), the Grassmann manifold of all nonoriented \( k \)-dimensional linear subspaces of the \( n \)-dimensional euclidean vector space. The fiber through \( \xi, \xi \in G^{k,n}, \) consists of all \( k \)-dimensional affine subspaces of \( E^n \) parallel to \( \xi \), and it is isomorphic to an orthogonal complement of \( \xi \) in \( E^n \), i.e. isomorphic to the \((n-k)\)-dimensional euclidean space. Hence the euclidean Laplace operator along the fibers defines an invariant differential operator \( \Box \) on \( G^{k,n} \) (for \( k = 0 \, \Box \) is the Laplace operator on the point space \( E^n \)).

For \( x \in G^{k,n}, l < k, G^{k,n}_x \) denotes the homogeneous space of all \( \xi \in G^{k,n} \) with \( x \subset \xi \). Taking an orthogonal complement of \( x \) in \( E^n \), \( G^{k,n}_x \) is isomorphic to the Grassmann manifold \( G^{k-l,n-l}_0 \).

Proposition 2.1 Let \( f \in C^\infty_c(E^n), n \geq 3, x \in G^{n-3,n} \), then

\[
(R^{n-3} f)(x) = -\frac{1}{4\pi^2} \int_{G^{n-1,n}} (\Box (R^{n-1} f))(\xi) \, d\xi,
\]

\((d\xi \text{ is the invariant volume density on } G^{n-1,n}_x, \text{ normalized by } \text{vol}(G^{n-1,n}_x) = \text{vol}(G^{2,3}_0) = 2\pi).\)

Proof: If \( \gamma \) is a regular parameterized smooth curve in \( X^n_x \), the second derivative of \( f \circ \gamma \) with respect to the given parameter writes

\[
(f \circ \gamma)'' = \text{hess}_f(\gamma', \gamma') + \text{grad}_f(\gamma'').
\]

This formula is basic for the following proof.

The case \( n = 3 \):

Let \( x \in E^3 \). Let \( B_x \) be the point bundle over \( G^{2,3}_x \), i.e. the totality of pairs \((y, \xi), y \in E^3, \xi \in G^{2,3}_x \) with \( y \in \xi \). Let \( \Gamma \) denote the foliation of \( B_x \) (except for the subset \( \{(y, \xi) \in B_x \mid y = x\} \) of measure zero) with 1-dimensional leaves as follows: the leaf passing through \((y, \xi)\) consists of all \((z, \eta) \in B_x\), where \( z \) runs through the circle \( \gamma \) with center \( x \) passing through \( y \).