ABSTRACT. In this paper we investigate the abstract angle measure for affine metric spaces. Common features and differences between orthogonal angles and angles with measure \( \neq 0 \) are examined. It turns out that an affine collineation which maps angles with a certain fixed measure \( \alpha \neq 0, 4 \) to angles with another fixed measure \( \beta \) is already a metric collineation in nearly all cases (fundamental theorem). An analogous result is stated for projective metric spaces. Some applications concerning minimal conditions for metric collineations are given.

1. Introduction

A metric vector space \((V, K, q)\) where \((V, K)\) is a vector space and \(q\) is a quadratic form on \(V\), gives rise to an affine metric and a projective metric space as well (cf. [1],[9],[10] for terminology and notation). The quadratic form \(q\) provides several different metric concepts such as orthogonality, congruence, conformity, and the angle measure. Depending on the chosen metric terminus one can define affine metric and projective metric spaces. If a concept is to contain all the metric information the quadratic form yields, then it must determine this form up to geometric equivalence, i.e., up to a factor \( \neq 0 \). As Schröder has shown in [9], this is provided by the planar conformity relation for (oriented) angles in the affine case and by the so-called metric product in the projective case. Hence, he defined an

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This article was written during the author’s stay as a postdoctoral fellow at the University of Saskatchewan in Saskatoon. The author would like to thank this institution, in particular Prof. Marshall, for the hospitality and the inspiring atmosphere.
affine metric space to be the affine space corresponding to \((V, K)\) together with the planar conformity relation \(\sim_q\) (in signs: \(A(V, K, \sim_q)\)). Analogously, a projective metric space is defined as the projective space corresponding to \((V, K)\) together with the metric product \(\mu_q\) (in signs: \((\Pi(V, K), \mu_q)\)). The fundamental theorems of affine metric and projective metric geometry give information on the isomorphisms of these spaces:

1. **Fundamental theorem of affine metric geometry:** (cf. [9]) Let \((A(V, K, \sim_q))\) and \((A(V', K', \sim_{q'})\) be affine metric spaces. If \(\varphi : A(V, K) \to A(V', K')\) is a collineation such that
   \[(g, h) \sim_q (k, l) \iff (\varphi(g), \varphi(h)) \sim_{q'}(\varphi(k), \varphi(l))\]
   for all angles \((g, h), (k, l)\) of \((A(V, K, \sim_q))\), then there exist a translation \(\tau\), a semilinear bijection \((\sigma_1, \sigma_2) : (V, K) \to (V', K')\), and a \(\lambda \in K' \setminus \{0\}\) such that \(\varphi = \tau \circ \sigma_1\) and \(q' = \lambda \sigma_2 \circ q\).

There is an analogous theorem for projective metric spaces (see [9]). In many cases we get exactly the same mappings if we consider those collineations which preserve orthogonality or the angle \(q\)-relation arising from the \(q\)-measure of angles (affine) resp. the Cayley-relation arising from the \(q\)-distance of points (projective) (cf. the “2. Fundamental theorem of affine metric (projective metric) geometry” in [9] and [10]). Hence, orthogonality seems to play a special role, i.e., the orthogonal angles appear to be somewhat special. We want to examine how particular this role really is (section 2). It turns out that orthogonal angles actually have some particular properties but as to the fundamental theorem, the preservance of orthogonality can be replaced almost always by the condition that angles of a fixed measure \(\alpha \neq 0, 4\) are mapped to angles of a fixed measure \(\beta\). To prove this we proceed as Schröder did in [9], i.e., we examine when a certain \(q\)-measure determines the underlying form up to a factor (section 3). To conclude this article we indicate some applications of our theorems concerning minimal conditions for metric collineations.

In the following we use the notation of [9]. Thus, for an affine metric space \((A(V, K, \sim_q))\) corresponding to the metric vector space \((V, K, q)\), and for lines \(g = A + KB, h = A + KC (A, B, C \in V)\) with \(q(B), q(C) \in K^* := K \setminus \{0\}\) let \(<_q (g, h) := f_q(B, C)^2q(B)^{-1}q(C)^{-1}\) be the \(q\)-measure of the (non-oriented) angle \(\{g, h\}\) where \(f_q\) is the bilinear form corresponding to \(q\) (for the motivation see [2]). If \((\Pi(V, K), \mu_q)\) is the corresponding projective metric space and \(KX, KY\) are points with \(q(X) \neq 0 \neq q(Y)\), then \(\delta_q(KX, KY) := f_q(X, Y)^2q(X)^{-1}q(Y)^{-1}\) is called the \(q\)-distance of \(KX\) and \(KY\). We denote by \(\text{ang}(V)\) (or \(\text{ang}(q)\)) the set of occurring angle measures \(\{\alpha \in K \mid \exists g, h \in G^r : <_q (g, h) = \alpha\}\) where \(G^r\) is the set of euclidean (or non-singular) lines (i.e., for \(g = R + KS\), \(g\) is euclidean iff \(q(S) \neq 0\)). By \(\text{dis}(V)\) (or \(\text{dis}(q)\)) we mean the set of occurring distances \(\{\alpha \in K \mid \exists X, Y \in V \setminus \ker q : \delta_q(KX, KY) = \alpha\}\) where \(\ker q := \{X \in V \mid q(X) = 0\}\).