TENSOR PRODUCT SURFACES OF EUCLIDEAN PLANE CURVES

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In honour of Professor KATSUMI NOMIZU on the occasion of his seventieth birthday

ABSTRACT

Recently B.Y. CHEN initiated the study of the tensor product immersion of two immersions of a given Riemannian manifold [31. In [6] the particular case of tensor product of two Euclidean plane curves was studied. The minimal one were classified, and necessary and sufficient conditions for such a tensor product to be totally real or complex or slant were established. In the present paper we study for tensor product of Euclidean plane curves the problem of B.Y. CHEN : to what extent do the properties of the tensor product immersion \( f \otimes h \) of two immersions \( f, h \) determines the immersions \( f, h \) ? [3]

Key words : tensor product immersions, minimal surfaces, Chen surfaces.

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1. Surfaces in \( \mathbb{E}^4 \)

Recall some results about surfaces \( M^2 \) in \( \mathbb{E}^4 \). For such a surface the set of focal points of the normal space \( T_m M^2 \) at a point \( m \) of \( M^2 \) is a conic \( F \) (the KOMMERELL conic). Sometimes another conic is used in the normal plane : the ellipse of normal curvature \( G \). \( F \) and \( G \) are polar w.r.t. a unit circle centered at \( m \) in the normal plane. We need for the following theorems the equation of \( F \). At \( m \in M^2 \) we choose an orthonormal frame \( \{ e_1, e_2, e_3, e_4 \} \) with \( e_1, e_2 \in T_m M^2 \), \( e_3, e_4 \in T_m^\perp M^2 \). Then we have the usual equations for this moving frame,

\[
\begin{align*}
\frac{dm}{dt} &= \omega^1 e_1 + \omega^2 e_2, \\
\frac{d\omega}{dt} &= \omega^i_j e_j, \\
\omega_i^j + \omega^j_i &= 0, \\
\omega_i^j &= \omega^{ik}_j, \\
\end{align*}
\]

where \( \omega^i_j \) and \( \omega_i^j \) are coefficients of the moving frame.

\( F \) is the set of \( p = x e_3 + y e_4 \) satisfying:

1. \( X^2 \left[ (h_{12}^3)^2 - h_{11}^3 h_{22}^3 \right] + XY \left[ 2h_{12}^3 h_{11}^4 h_{12}^4 - h_{11}^4 h_{22}^4 \right] + Y^2 \left[ (h_{12}^4)^2 - h_{11}^4 h_{22}^4 \right] \]

or

2. \( (X h_{12}^3 + Y h_{12}^4)^2 - (X h_{11}^3 + Y h_{11}^4 - 1) (X h_{22}^3 + Y h_{22}^4 - 1) = 0. \)

We consider now characterizations of special surfaces of \( \mathbb{E}^4 \) using properties of \( F \).
a) **Surfaces** $M^2$ **with null Gaussian curvature** $K$.

$K$ is defined by $Dw^2_1 = K w^1 \wedge \omega^2$, and then $K = - (h_{12}^3)^2 - (h_{12}^4)^2 + h_{11}^3 h_{22}^2 + h_{11}^4 h_{22}^4$.

We see from (1) that $M^2$ have null Gaussian curvature if and only if $F$ is an orthogonal hyperbola.

b) **Surfaces with null normal curvature** $K_N$.

$K_N$ is defined by $Dp^4_3 = - K_N w^1 \wedge \omega^2$ and then $K_N = h_{12}^3 (h_{22}^3 h_{11}^4) + h_{12}^4 (h_{11}^3 - h_{22}^3)$.

From equation (2) we see that $F$ is the union of two straight lines if and only if $K_F = 0$.

c) **Ruled surfaces** of $E^4$.

If $M^2$ is a ruled surface of $E^4$ generated by straight lines $\Delta$, if we choose at any point $m \in M^2$ the tangent vector on $\Delta$, then $d_e = 0$ for $\omega^2 = 0$ and then $h_{11}^3 = h_{11}^4 = 0$. We see from (1), that in this case $F$ is a parabola.

d) **Minimal surfaces**

They satisfy $H = 0$, where $H$ is the mean curvature vector

$$H = \frac{1}{2} \left\{ \left( h_{11}^3 + h_{22}^3 \right) e_3 + \left( h_{11}^4 + h_{22}^4 \right) e_4 \right\}$$

Then from (1) $M^2$ is a minimal surface if and only if the center of $F$ is $m$.

e) **CHEN surfaces**

They are defined by $a(H) = 0$ where $a(H)$ is the allied mean curvature vector.

We see in [7] that $M^2$ is a CHEN surface if and only if $H$ is an axis of $F$. If the equation of $F$ is $Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$, the condition is

$$B(E^2 - D^2) + DE(A - C) = 0.$$ 

f) **Pseudo-umbilical surfaces**

They are defined by $A_H = \lambda I$ where $A_H$ is the shape operator associated with the normal vector $\frac{H}{|H|}$.

It is easy to show that $M^2$ is pseudo-umbilical if and only if $F$ is the union of two straight lines with $H$ as axis of symmetry.

g) **Surfaces for which $F$ is a circle** (studied by R. CALAPSO [2]) or $G$ is a circle (studied by O. BORUVKA [1]).