A Characterization of Bol left loops

Dedicated to Prof. Dr. Dr. h.c. Herbert Zeitler on the occasion of his 80th birthday

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Abstract

The relationship between the Bol identity, the so-called Left Loop Property LLP and the Left Inverse Property in any left quasigroup is determined. Counterexamples are given whenever two properties are not equivalent. It is shown that a principal isotope of a LLP quasigroup is a left Bol loop. In any LLP left quasigroup the existence of a right identity element is equivalent to the right division.

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1 Introduction

In order to describe the algebraic structure of the relativistic velocity addition, A.A. Ungar introduced the notion of a gyrogroup (first called weakly associative group) [13, 14]. It turns out that gyrogroups coincide with K-loops (cf. [6]). K-loops describe the additive structure of neardomains which H. Karzel introduced in [4] in order to describe sharply 2-transitive groups. Both notions are equivalent to the notion of Bol loops (cf. [1]) with the additional property that the inner mappings are automorphisms of the loop (sometimes called Bruck loops or Bol loops with Bruck identity (cf. [5, 7, 10, 11]). This connections suggest to investigate the different axioms which define Bol loops also for the more general case of quasigroups. This gives an understanding of the importance of a left or right identity in connection with different equations.
We recall some notations (cf. [7]). A binary system \((Q, \cdot)\) is said to be a left quasigroup if and only if for any two elements \(a, b \in Q\) there is a unique element \(x = a \setminus b \in Q\) with \(a \cdot x = b\). A left quasigroup is called a left loop if there exists a right identity element \(e \in Q\) with \(a \cdot e = a\) for all \(a \in Q\). (In some other papers is what we call here left loop denoted as right loop). Likewise one defines a right quasigroup and a right loop. A quasigroup (loop, identity element) is a left and right quasigroup (left and right loop, left and right identity element).

For a left quasigroup \((Q, \cdot)\) and \(x \in Q\) we denote by 
\[
e_x \quad \text{the unique element of } Q \text{ with } x \cdot e_x = x
\]
\[
x^r \quad \text{the unique element of } Q \text{ with } x \cdot x^r = e_x.
\]

The property

\[(LIP) \quad x \cdot x^r y = y\]

for any \(x, y \in Q\) we call the left inverse property (cf. Lemma 2.4 with Remark for equivalent definitions). For a left quasigroup \((Q, \cdot)\) and \(a, b \in Q\) the mapping \(L_a : Q \to Q; x \to a \cdot x\) is a permutation and \(l_{a,b} := L_{a,b}^{-1}L_b L_a\) is called a left inner mapping. As easily seen \(a \cdot bx = ab \cdot l_{a,b}(x)\). A. Ungar uses for the definition of a gyrogroup the so called left loop property

\[(LLP) \quad l_{a,b} = l_{a, b, b}\]

which is equivalent by [3], Lemma 2.1, to the equation

\[(LLP') \quad a(b \setminus a) \cdot c = a(b \setminus (ac))\]

For \(a, b, c \in Q\) the equation

\[(LBI) \quad a(b \cdot ac) = (a \cdot ba)c\]

is called the (left) Bol identity. Using the left inner mappings we compute \(a(b \cdot ac) = (a \cdot ba)l_{a,ba}l_{b,a}(c)\) and LBI is equivalent to

\[(LBI') \quad l_{b,a}^{-1} = l_{a,ba}\]

It is known that a (left) Bol left quasigroup \((Q, \cdot)\) satisfies the Left Inverse Property LIP and has a right identity if and only if \((Q, \cdot)\) is a quasigroup (cf. [9, 12] and Lemma 2.2). It is also proved that a loop satisfies the (left) Bol identity LBI if and only if it satisfies the left loop property LLP (cf. [3, 11]). It is our concern in this note to determine the relationship between LBI, LLP and LIP for any left quasigroup and we generalize known results to left quasigroups. We give counterexamples whenever two properties are not equivalent (section 2). In section 3 an isotopy theory for LLP left quasigroups is initiated. It turns out that a principal isotope of a LLP quasigroup is a left Bol loop (Theorem 3.1). Further a loop-theoretical construction of LLP left quasigroups (loops) from left Bol left quasigroups