Differential Operator Rings for Which Homogeneous Functions Are Linear

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Abstract

Let $\mathcal{E}$ be the collection of those rings $S$ such that, for every $S$-module $V$ and every homogeneous function $f: V \to V$, $f(sv) = sf(v)$, $s \in S$, $v \in V$, $f$ is linear on $V$. In this paper we characterize those Stanley-Reisner rings, $R$, such that $D(R)$ is in $\mathcal{E}$.

I Introduction

Let $R$ be a ring with identity and $V$ a unitary $R$-module. Let $M_R(V) := \{f: V \to V| f(rv) = rf(v), \forall r \in R, \forall v \in V\}$, the near-ring of homogeneous functions determined by the pair $(R, V)$. As usual $\text{End}_R(V)$ denotes the ring of $R$-endomorphisms of $V$. By definition, $\text{End}_R(V) \subseteq M_R(V)$. The problem of characterizing those rings $R$ such that $M_R(V) = \text{End}_R(V)$ for each $R$-module $V$ was initiated in [4] where the collection of such rings was denoted by $\mathcal{E}$, i.e.,

$$\mathcal{E} := \{\text{rings } R| M_R(V) = \text{End}_R(V), \forall V \in R\text{-mod}\}.$$ 

It was found in [4], Theorem II.8, that for any ring $S$ with identity, the matrix ring $M_n(S)$ belongs to $\mathcal{E}$ whenever $n \geq 2$. Other rings in $\mathcal{E}$ were identified and the Artinian rings in $\mathcal{E}$ were characterized. However, no characterization of $\mathcal{E}$ was obtained, and moreover, to the author’s knowledge, no characterization has yet been obtained. Recently [3], Peter Fuchs showed that a simple Noetherian ring, $S$, is in $\mathcal{E}$ if and only if $S$ is not a domain.

In this paper we identify another large class of rings which are in $\mathcal{E}$, namely rings of differential operators of certain Stanley-Reisner rings. In fact, we characterize those Stanley-Reisner rings, $R$, such that $D(R)$ is in $\mathcal{E}$.

2000 Mathematics Subject Classification. Primary 13N10, 13F55, 16S50

Keywords: Differential operator rings; near-rings of homogeneous functions; Stanley-Reisner rings.
In the next section we present the pertinent definitions and background material on Stanley-Reisner rings and on rings of differential operators needed in the sequel. In the third section we present our main results and in the final section we mention some other classes of rings whose rings of differential operators are in \( \mathcal{E} \).

Conventions and Notation. Throughout the paper, all rings have an identity, all modules are unital and all ring homomorphisms are identity preserving.

For any subset \( T \) of a ring \( R \), \( T^* = T - \{ 0 \} \) and \( (T) \) denotes the (two-sided) ideal of \( R \) generated by \( T \).

Also, \( k \) will always denote a field and \( k[x_1, \ldots, x_N] \) is the ring of polynomials in the \( N \) indeterminates \( x_1, \ldots, x_N \). We let \( X = \{ x_1, \ldots, x_N \} \) and often denote the polynomial ring \( k[x_1, \ldots, x_N] \) by \( k[X] \).

II Background Concepts

II.A Stanley-Reisner Rings

In this first subsection we present basic concepts of Stanley-Reisner rings which we will use in the remainder of the paper. Most of these results are standard and will be stated without proof.

A monomial ideal \( I \) of \( k[X] \) is an ideal generated by monomials in \( x_1, \ldots, x_N \). When \( I \) is a monomial ideal, the quotient ring \( k[X]/I \) is called a monomial ring. For further information on monomial ideals and monomial rings we refer the reader to the recent monograph of Villarreal, [12]. We assume henceforth for simplicity that \( x_i \notin I, \ i = 1,2,\ldots,N \).

Recall that a commutative ring is reduced if it does not contain any nonzero nilpotent elements. A reduced monomial ring is called a Stanley-Reisner ring. We recall further that Stanley-Reisner rings are face rings of simplicial complexes and that there has been a great deal of transfer of information between simplicial complexes and their associated Stanley-Reisner rings. See [1], [10], or [12].

Now let \( k[X]/I \) be a Stanley-Reisner ring. Since this monomial ring is reduced, \( I \) is generated by square-free monomials which in turn means that \( I \) is a radical ideal and that \( I \) is the intersection of its minimal primes, [12]. Suppose \( I = (m_1,m_2,\ldots,m_t) \) where the \( m_i \) are square-free monomials in \( k[X] \) and \( \{ m_1,m_2,\ldots,m_t \} \) is a minimal generating set for \( I \). We use the following:

i) if \( m_i = x_{i_1}x_{i_2}\ldots x_{i_{n_i}} \) then \( \text{Supp}(m_i) = \{ x_{i_1},x_{i_2},\ldots,x_{i_{n_i}} \} \);

ii) \( \text{Supp}(I) = \bigcup_{i=1}^{t} \text{Supp}(m_i) \). (Since \( \{ m_1,\ldots,m_t \} \) is a minimal generating set of square-free monomials we see that \( \text{Supp}(I) \) depends only on \( I \).)

Theorem A ([11]). The minimal primes over a monomial ideal \( I = (m_1,\ldots,m_t) \) are all of the form \( P = (x_{i_1},\ldots,x_{i_j}) \) where

1) \( x_{i_t} \in \text{Supp}(m_i) \) for some \( i = 1,2,\ldots,t \);

2) for each \( x_{i_t} \), there exists some \( m_j \) such that \( x_{i_t} \in \text{Supp}(m_j) \) but no other \( x_{i_s} \in \text{Supp}(m_j) \).