ON HYERS—ULAM STABILITY OF HOSZSÚ’S FUNCTIONAL EQUATION

COSTANZA BORELLI

Dedicated with affection and appreciation to Professor János Aczél on his seventieth birthday.

Abstract. In this paper the Hyers—Ulam stability of the Hosszú functional equation is proved.

1. Introduction

In this paper we treat the problem concerning the stability, in the sense of Hyers—Ulam, of the Hosszú functional equation on the field of the real numbers.

A function $f : \mathbb{R} \to \mathbb{R}$ satisfies the Hosszú functional equation if and only if for all $x, y \in \mathbb{R}$

$$(H) \quad Hf(x, y) := f(x + y - xy) + f(xy) - f(x) - f(y) = 0.$$  

It is well known that $f$ satisfies the Hosszú equation $(H)$ if and only if it has the form

$$(1) \quad f(x) = a(x) + b$$

where $a : \mathbb{R} \to \mathbb{R}$ is an additive function and $b \in \mathbb{R}$ (see [4]).

Now we consider the inequality

$$(HI) \quad |Hf(x, y)| \leq \delta$$

and we prove the following.

Theorem. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $(HI)$. There exists an additive function $a : \mathbb{R} \to \mathbb{R}$ such that the difference $f - a$ is bounded if and only if the even part $h$ of $f$ satisfies $|Hh(x, y)| \leq \epsilon$ for some positive $\epsilon$.  

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2. PROOF OF THEOREM

The proof of our result is obtained as a consequence of the following lemmas. The functions $g$ and $h$ throughout this paper represent the odd and even part of $f$ respectively.

Lemma 1. If $|Hh(x,y)| \leq \varepsilon$ for all $x, y \in \mathbb{R}$, then $|h(x)| \leq 2\varepsilon + |h(1)|$ for all $x \in \mathbb{R}$.

Proof. We have $|Hh(-x,-y)| \leq \varepsilon$; since $h$ is even we obtain

$$|h(-x-y-xy) + h(xy) - h(x) - h(y)| \leq \varepsilon.$$ 

Thus

$$|h(x+y) - h(-x-y-xy)| = |h(x+y) + h(xy) - h(x) - h(y) - h(-x-y-xy) - h(xy) + h(x) + h(y)| \leq |Hh(x,y)| + |Hh(-x,-y)| \leq 2\varepsilon$$

and by setting $y = -1$ we get $|h(2x-1) - h(1)| \leq 2\varepsilon$; since $2x-1$ spans the whole $\mathbb{R}$, we are finished. □

Lemma 2. Define $\mu := \delta + \varepsilon$ and $\gamma := \max\{2\mu, |g(2) - 2g(1)|\}$. Then $g(x) = p(x) + s(x)$, where $p$ and $s$ are odd functions, $p$ satisfies the functional equation $p(2x) = 2p(x)$ and $|s(x)| \leq \gamma$ for all $x \in \mathbb{R}$.

Proof. Obviously it is $|Hg(x,y)| \leq \mu$ and $|Hg(x,-y)| \leq \mu$. Since $g$ is odd we obtain $|g(x+y+xy) - g(xy) - g(x) + g(y)| \leq \mu$ and so

$$|g(x+y-xy) + g(x-y+xy) - 2g(x)| = |Hg(x,y)| + |Hg(x,-y)| \leq 2\mu.$$ 

By taking $y = x(1-x)^{-1}, x \in \mathbb{R} \setminus \{1\}$, we obtain

$$|g(2x) + g(0) - 2g(x)| = |g(2x) - 2g(x)| \leq 2\mu, \quad \text{for } \quad x \in \mathbb{R} \setminus \{1\}$$

($g(0) = 0$ since $g$ is odd) and so

$$|g(2x) - 2g(x)| \leq \gamma \quad \text{for all } \quad x \in \mathbb{R}.$$ 

By using well-known results on stability (see for instance [2]) we can conclude that

$$p(x) := \lim_{n \to \infty} 2^{-n}g(2^n x)$$

exists for every $x \in \mathbb{R}, p(2x) = 2p(x)$, $p$ is odd and $|g(x) - p(x)| = |s(x)| \leq \gamma$ for all $x \in \mathbb{R}$. □