AN ASYMPTOTIC FORMULA FOR THE ITERATES OF A FUNCTION

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ABSTRACT:

Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$ and $D$ an open set of $\mathbb{K}$ containing 0 and starlike with respect to 0 (i.e. an open interval containing 0 in the case $\mathbb{K} = \mathbb{R}$). If $f : D \to \mathbb{K}$ is a continuous function with fixed point 0, then under certain conditions stated below we can prove for the $kn$-th iterates of $f$ the following asymptotic formula:

$$f^{(kn)} \left( \frac{x}{n} \right) = \sum_{i=1}^{r} \frac{1}{(nk)^i} f_i(kx) + o \left( \frac{1}{n^r} \right)$$  \hspace{1cm} (1)

for $n \to \infty$, $k$, $n$ and $r$ being positive integers and $x$ close enough to 0. The functions $f_i$ are continuous and uniquely determined by $f$.

In particular (1) holds for any function holomorphic on a neighbourhood of zero, having a convergent power series expansion of the form

$$f(z) = z + a_2 z^2 + \cdots = \sum_{j=1}^{\infty} a_j z^j, \hspace{0.5cm} a_j \in \mathbb{C}, \hspace{0.5cm} a_1 = 1,$$

and for any integers $k$, $r$ with $r > 0$.

1. INTRODUCTION:

In a recent paper [1], L. Berg investigated the asymptotic properties of the real solutions of the translation equation

$$F(z, s + t) = F(F(z, s), t). \hspace{1cm} (T)$$

L. Berg obtained under conditions analogous to those used in the present paper, amongst other results, the following asymptotic formula for the solutions of (T):

$$F(z/n, nt) = (z/n) f_1(zt) + (z/n)^2 f_2(zt) + (z/n)^3 f_3(zt) + o(n^{-3}) \hspace{1cm} (\star)$$

For integers $k$, a solution $F(z, k)$ of (T) is exactly the $k$-th iterate of the function $f(z) = F(z, 1)$, i.e. $F(z, k) = f^{(k)}(z)$, therefore the formula (1) is a generalization of the formula of L. Berg to those functions which need not to be embeddable in a one parameter group (see Targonski [4] and section 4, this paper). However, the proofs given here are in a more or less close analogy to those of the paper [1] of L. Berg.

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2. The asymptotic formula:

We will prove the following.

**Theorem 1:** Let \( \mathbb{K} \) be either \( \mathbb{R} \) or \( \mathbb{C} \) and \( D \) and \( D' \) open sets of \( \mathbb{K} \) containing 0 and starlike with respect to 0, \( D' \subseteq D \). Let \( f : D \to \mathbb{K} \) be continuous on \( D \) with \( f(0) = 0 \) and suppose that for all natural \( k \), smaller or equal to a fixed \( k_0 \),

\[
\lim_{n \to \infty} n f^{(kn)}(z/n) \tag{2}
\]

exists uniformly on \( D' \). Then there is a unique function \( f_1 \) defined and continuous on \( D' \) and with \( f_1(0) = 0 \) such that

\[
f^{(kn)} \left( \frac{x}{n} \right) = \frac{1}{nk} f_1(kx) + o \left( \frac{1}{n} \right) \tag{3}
\]

for \( n \to \infty \), \( x, kx \in D' \) and \( 1 \leq k \leq k_0 \).

**Proof:** We may define \( \varphi_k(z) = \lim_{n \to \infty} n f^{(kn)}(z/n) \) which is a continuous function on \( D' \). For a positive integer \( m \) let further be \( g_n(z, m) := n \cdot f^{(mn)}(z/n) \), hence \( \lim_{n \to \infty} g_n(z, m) = \varphi_m(z) \). With \( A_{mn}(z) := mn \cdot f^{(mn)}(z/mn) \) follows \( A_{mn}(z) = mg_m(z/m, m) \) and \( A_{mn}(z) = ng_m(z/n, n) \). Using some arguments of elementary analysis one gets:

\[
\lim_{n \to \infty} A_{mn}(z) = \lim_{m \to \infty} A_{mn}(z).
\]

Hence for \( m, n \) (less or equal \( k_0 \))

\[
m\varphi_m(z/m) = n\varphi_n(z/n).
\]

Putting \( z = mx \) and \( n = 1 \), this yields \( \varphi_m(x) = (1/m)\varphi_1(mx) \). With the definition \( f_1(x) = \varphi_1(x) \) we get a function defined and continuous on \( D' \). Now

\[
\lim_{n \to \infty} n \left( f^{(kn)} \left( \frac{x}{n} \right) - \frac{1}{nk} f_1(kx) \right) = \lim_{n \to \infty} n f^{(kn)} \left( \frac{x}{n} \right) - \varphi_k(x) = 0.
\]

This proves formula (3). Obviously, \( f_1(0) = 0 \) and the function \( f_1 \) is uniquely defined by the existence of (2). \( \square \)

Now we proceed by induction. We may mention that the set \( D'' = \{ x | kx \in D' \} \) is again open and starlike with respect to 0, and \( D'' \subseteq D' \subseteq D \). For the sake of simplicity we will take for the region of uniform convergence of the functions in the following theorem the same denomination as in Theorem 1.

**Theorem 2:** Let \( f, D, D' \) and \( k_0 \) be given as in Theorem 1 and suppose that for continuous functions \( f_1, \ldots, f_r \) with fixed point 0 and for \( 0 < k \leq k_0 \) the limit

\[
\lim_{n \to \infty} n^{r+1} \left( f^{(kn)} \left( \frac{x}{n} \right) - \sum_{i=1}^{r} \frac{1}{(nk)^i} f_i(kx) \right) \tag{4}
\]

exists uniformly on \( D' \). Then there is a unique continuous function \( f_{r+1} \) with fixed point 0 such that

\[
f^{(kn)} \left( \frac{x}{n} \right) = \sum_{i=1}^{r+1} \frac{1}{(nk)^i} f_i(kx) + o \left( \frac{1}{n^{r+1}} \right), \tag{5}
\]

for \( n \to \infty \), and \( x, kx \in D' \) and \( 1 \leq k \leq k_0 \).