ON K-LOOPS OF FINITE ORDER

To the memory of Hans Zassenhaus

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Abstract. In this note we undertake an axiomatic investigation of K-loops (or gyro-groups, as A.A. Ungar used to name them) and provide new construction methods for finite K-loops. It is shown how, more or less, the axioms are independent from each other. Especially (K6) is independent as A.A. Ungar already had conjectured. We begin with right loops \((L, \oplus)\) and add step by step further properties. So the connection between K-loops, Bol-loops, Bruck-loops and the homogeneous loops of Kikkawa became clear. The smallest examples of proper K-loops possess 8 elements; there are exactly 3 non-isomorphic of these. At last it is shown that one gets quite naturally a Frobenius-group as a quasidirect product of a K-loop \((L, \oplus)\) and a group \(D\) of automorphisms of \((L, \oplus)\) if \(D\) is fixed point free except from 0.

1. INTRODUCTION

In order to describe sharply 2-transitive groups, H. Karzel introduced in [9] the notion of a neardomain \((F, \oplus, \cdot)(\text{cf. } [33])\). The crucial difficulty of a neardomain is the additive structure \((F, \oplus)\), which need not be associative. A neardomain \((F, \oplus, \cdot)\) with an associative addition is already a nearfield. In many notes neardomains are investigated \((\text{cf. } [10, 13, 14, 15, 16, 17, 19, 33, 34])\), but until today no example of a proper neardomain is known. To obtain partial results, W. Kerby and H. Wefelscheid considered separately the additive structure \((F, \oplus)\) and called such loops K-loops (see definition in section 2), but since they could not find a proper example of a K-loop no further theoretical investigations were done. The interest on K-loops has been revived grown since A. A. Ungar 1988 has found a famous physical example.

A.A. Ungar investigated the relativistic addition \(\oplus\) of the velocities \(\mathbb{R}_{\mathbb{C}}^3 := \{ v \in \mathbb{R}^3 : |v| < c \} \)
He showed that \((\mathbb{R}^3_\infty, \circ)\) is a non-associative and non-commutative loop with characteristic automorphisms, the so-called Thomas rotations. He proved that for any two velocities \(a, b \in \mathbb{R}^3_\infty\) there is a Thomas rotation \(\delta_{a,b}\) fulfilling \(a \circ (b \circ x) = (a \circ b) \circ \delta_{a,b}(x)\) (cf. \([29, 30, 31]\)). H. Wefelscheid recognized then that \((\mathbb{R}^3_\infty, \circ)\) is a K-loop.

But there is also a close connection between K-loops and Bruck loops, which is discovered first by G. Kist [19]. Bruck loops are Bol loops satisfying the automorphic inverse property \((K5)\) (cf. section 2). Bol loops are introduced by G. Bol in 1937 in order to coordinatize webs (cf. \([1])\), and are investigated in later years in many papers \([2 \text{ to } 6, 12, 25 \text{ to } 28]\). The examples in Bol's paper are due to Zassenhaus. Bol also seemed having had difficulties in finding proper examples. In \([19, \S 1.3]\) G. Kist remarks, that already from Lemma 6 of G. Glauberman [6] one can deduce that every finite Bruck loop of odd order satisfies \((K3)\). As a generalisation it is proved in \([21, \text{ Theorem 1}]\) that every Bruck loop with no element of order 2 is a K-loop. Hence many examples of Bruck loops turns out to be examples of K-loops (cf. \([12, 25]\)). Other examples are given by A. Kreuzer in \([21, 22]\).

In this note we consider right loops with the axiom \((K3)\), not necessarily finite, and add step by step the other axioms of a K-loop if necessary for stronger results. In section 2 a comprehensive list of properties of right loops is presented. Let \(D\) denote the group of automorphisms of a right loop \((L, \circ)\) with \((K3)\) and \((K4)\), generated by the maps \(\delta_{a,b}\). Then one can introduce on \(G := L \times D\) a group operation (cf. \((3.5)\)). In section 3 we consider the inverse of that process, starting with a group \(G\) and an exact decomposition \(G = K \cdot A\). In section 4 a new construction method for a loop operation on the set \(G \times H\) for commutative groups \(G, H\) is given. In section 5 and 6 we consider finite examples of K-loops constructed with that method. If any automorphism \(\delta \in D \setminus \{\text{id}\}\) of a K-loop \((L, \circ)\) has only the fixed point 0, then \(L\) determines a Frobenius group. In the finite case that fact implies that for proper finite K-loops there always does exist an automorphism \(\tau \in D \setminus \{\text{id}\}\) having fixed point distinct from 0 (cf. section 7).

2. DEFINITIONS AND PROPERTIES OF LOOPS

Let \(L\) be a set with a binary operation \(\circ\). We call \((L, \circ)\) a right loop if \((K1r), (K2)\) are valid, and a loop, if \((K1r), (K1l)\) and \((K2)\) are valid for all \(a, b \in L\).