ON A CONJECTURE OF J. WEIDMANN

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ABSTRACT. Consider the Sturm-Liouville differential expression $l(y) = -y'' + q(x)y$ on an interval $(a, b)$ and assume that $l$ is in the limit point case at $b$. Fix $c \in (a, b)$ and let $L_a$, $L_b$ be self-adjoint realizations of $l$ in $L^2(a, b)$, $L^2(c, b)$ respectively. If $L_b$ has purely absolutely continuous spectrum in an interval $J$ and if the spectral function $\rho_b$ of $L_b$ satisfies some mild growth conditions then the spectrum of $L$ in $J$ is shown to be purely absolutely continuous, too. Our result confirms a conjecture of J. Weidmann (1982). It had been shown by del Rio Castillo (1988) that in Weidmann's original formulation this conjecture is false.

1. INTRODUCTION.

In [12] J. Weidmann formulated the following conjecture:
(C) Let $l$ denote a formally self-adjoint differential expression on $(a, b)$ and $L$ a self-adjoint realization. For some $c \in (a, b)$ let $L_a$ and $L_b$ be self-adjoint realizations of $l$ on $(a, c)$ resp. $(c, b)$. If $L_a$ or $L_b$ have purely absolutely continuous spectrum in an interval $[\mu_1, \mu_2]$ then the same is true for $L$.

It has been shown by del Rio Castillo [3] that in this form the conjecture
is false. Even the stronger hypothesis that each self-adjoint realization $L_a$ (or $L_b$) has purely absolutely continuous spectrum in $[\mu_1, \mu_2]$ cannot guarantee that the spectrum of $L$ is purely absolutely continuous, too! However, it could be shown in [7] that the conclusions of the above conjecture hold if some additional "growth conditions" are imposed on the spectral function $\rho_b$ of $L_b$.

The aim of the present paper is to weaken the assumptions of [7] as far as possible. We are going to show for a Sturm-Liouville operator $l$ that two coupled $L^p$-conditions on the derivative $\rho'_{b}$ of $\rho_{b}$ are sufficient to imply the absolute continuity of the spectral function $\rho$ of $L$. Moreover we obtain additional smoothness properties of $\rho$ regardless of the behaviour of the differential expression $l$ at the endpoint $a$. It is of some interest also that the conditions we impose on $\rho_{b}$ can be translated into conditions on the boundary behaviour of the Titchmarsh-Weyl coefficient $m_{b}$ of the operator $L_{b}$ since from a practical point of view the coefficient $m_{b}$ is more easily accessible than the corresponding spectral function $\rho_{b}$.

2. STATEMENT OF THE RESULTS.

Let us consider the Sturm-Liouville differential expression

\begin{equation}
(2.1) \quad l(y) = -y'' + q(x)y, \quad x \in (a, b), \quad -\infty < a < b < +\infty,
\end{equation}

where $q$ is real-valued and locally integrable. We shall assume that the limit-point case occurs at $b$. The endpoint $a$ may be limit-circle or limit point. Now fix $c \in (a, b)$ and let $L, L_{a}, L_{b}$ be self-adjoint realizations of $l$ in $L^2(a, b), L^2(a, c), L^2(c, b)$ respectively such that the boundary conditions of $L, L_{a}$ coincide at $a$ and the boundary conditions of $L_{a}, L_{b}$ coincide at $c$. (There is no boundary condition at the point $b$).

Following [7] e.g. (see also [2], [6]) there exist two functions $m_{a}, m_{b}$ (the Titchmarsh-Weyl coefficients of $L_{a}, L_{b}$), analytic in the upper half-plane $C_{+} = \{ \lambda \in \mathbb{C} \mid \text{Im}\lambda > 0 \}$ such that

\begin{equation}
(2.2) \quad \text{Im}m_{a}(\lambda) < 0, \quad \text{Im}m_{b}(\lambda) > 0 \quad \text{on} \quad C_{+}.
\end{equation}

The spectral function $\rho_{b}$ of $L_{b}$ is a nondecreasing real-valued function.