REGULAR SETS AND GEOMETRIC GROUPS

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If \( G \) is a permutation group acting on a set \( \Omega \), a subset \( \Lambda \) of \( \Omega \) is called a regular set for \( G \) if the set-stabilizer of \( \Lambda \) in \( G \) is the identity subgroup. We show here that the projective and affine semi-linear groups acting in the natural way as permutation groups on their respective finite geometries, have, in general, for all finite dimensions and all finite fields, regular sets of points. The exceptions to this are found, and an extension of the results to infinite fields is discussed.

1. INTRODUCTION

In a geometry a set of elements is regular if only the identity automorphism leaves these elements invariant as a set. In this paper we shall show that there are such regular sets in affine and projective spaces over finite fields of at least three elements. Exceptions do occur for some low-dimensional spaces over small fields. These results are given in Theorems 3.1 and 4.1. For permutation groups the notion of a regular set is similar: a set is regular for a group \( G \) if only the identity in \( G \) leaves the set invariant. Cameron, Neumann and Saxl [2] showed that all finite primitive groups of sufficiently large degree, not containing the alternating group, have regular sets. From our results here it is immediate that any collineation group of an affine or projective space over a finite field of at least three elements, has regular sets, apart from some finite number of exceptions. Recently, Dalla Volta [4] has shown that the same is true for the field of two elements. There are related results on regular sets in [6] and [11].

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A permutation group is geometric if there is an incidence structure on the same set of points such that the group is the full automorphism group of this structure. This yields a concept related to regular sets and is due to Betten [1]. In Section 5 of this paper we show that - apart from some exceptions - all collineation groups of affine or projective spaces are geometric (Theorem 5.1).

In [8], [9], [12] and [13] a permutation group $G$ is called $k$-closed if the largest group having the same orbits as $G$ in the natural action on the system of $k$-element subsets coincides with $G$. The results of this paper are used to show that a permutation group of odd degree containing an elementary abelian normal and transitive subgroup is $k$-closed and bounds for $k$ are given by logarithmic functions of the degree. The results in this paper are independent of the classification theorem of finite simple groups.

The paper is organized as follows: Section 2 deals with definitions and notation, with some preliminary lemmas; Sections 3 and 4 deal with regular sets for the projective and affine cases respectively, leading to the main theorems on regular sets Theorems 3.1 and 4.1; Section 5 establishes $k$-geometric properties of subgroups of the semi-linear groups; Section 6 deals with analogous results for infinite fields.

2. PRELIMINARIES

The terminology and notation used is mostly that of Dembowski [5] and Wielandt [15], with exceptions as noted below. The symmetric and alternating groups on a set $\Omega$ are denoted by $\text{Sym}(\Omega)$ and $\text{Alt}(\Omega)$ respectively, or $S_n$ and $A_n$ if $|\Omega| = n$. If $G \leq \text{Sym}(\Omega)$, and $\Delta \subseteq \Omega$, then $G_{\{\Delta\}}$ denotes the set stabilizer of $\Delta$ and $G(\Delta)$ denotes the pointwise stabilizer.

We will refer to $G_{\{\Delta\}}/G(\Delta)$ as the restriction of $G$ to $\Delta$, written as $G^\Delta$. For $k \leq n$, $G$ acts in a natural way on the set $\Omega^{\{k\}}$ of all $k$-element subsets of $\Omega$. The $k$-closure $G^{\{k\}}$ of $G$ is the largest subgroup of $\text{Sym}(\Omega)$ having the same orbits on $\Omega^{\{k\}}$ as $G$. $G$ is $k$-closed if $G = G^{\{k\}}$. When speaking of k-closure we will always assume that $k \leq \frac{n}{2}$. Members of $\Omega^{\{k\}}$ will be called $k$-sets. For $\Delta \subseteq \Omega$, the set of all images of $\Delta$ under $G$ is denoted by $\Delta^G$. All sets and groups are