GEOMETRIC INEQUALITIES FOR POISSON PROCESSES OF CONVEX BODIES AND CYLINDERS

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1. Introduction

The aim of the present paper is the investigation of some extremum problems and inequalities of isoperimetric type for certain random systems of convex sets. These considerations extend and complement recent work of Wieacker [36].

Let $\mathcal{K}^d$ denote the space of convex bodies (non-empty, compact, convex subsets) in $d$-dimensional Euclidean space $\mathbb{R}^d$, as usual equipped with the Hausdorff metric and the induced topology and Borel structure. Let $Y$ be a random point process in $\mathcal{K}^d$. We assume that its intensity measure $\Theta$ is translation invariant and satisfies $\Theta(\{K \in \mathcal{K}^d : K \cap C \neq \emptyset\}) < \infty$ for some fixed body $C$ with interior points. Any realization of such a process consists of a system of convex bodies in $\mathbb{R}^d$ with the property that a given convex body hits almost surely only finitely many bodies of the system. With such a system one can associate various intuitive quantities to measure the geometric behaviour and the "denseness". Wieacker [36] studied two such quantities which are particularly simple and tractable. The first of these is the intersection point.
density $D_0(Y)$, which can be defined as follows. For a Borel set $A \subset \mathbb{R}^d$ with positive Lebesgue measure $\lambda^d(A)$, consider the points in $A$ which arise as intersection points of any $d$ boundaries of bodies in the system. Their number is a random variable, and its expectation, divided by $\lambda^d(A)$, is independent of $A$ and defines the intersection point density $D_0(Y)$ of the process $Y$. The second geometric notion which measures in some sense the denseness of the system, is the visible volume: Let $X_Y$ be the union of the convex bodies belonging to the random system. For a point $x \notin X_Y$, let $S_x$ be the subset of $\mathbb{R}^d \setminus X_Y$ which is visible from $x$, if $X_Y$ is considered as opaque. Then the conditional expectation of the random variable $\lambda^d(S_x)$ under the condition $x \notin X_Y$ is independent of $x$ and defines the visible volume $V_s(Y)$ of the process $Y$.

Both numbers, the intersection point density and the visible volume, measure in some sense the denseness of the process $Y$, but in opposite directions: If the intersection point density is large, one would intuitively expect that the visible volume is small. For Poisson processes $Y$, this can be made surprisingly precise. In fact, Wieacker [36] discovered that

\begin{equation}
D_0(Y)V_s(Y) \leq d \lambda^2_d.
\end{equation}

where $\lambda_d$ denotes the volume of the $d$-dimensional unit ball. If the Poisson process $Y$ is isotropic (which means that the intensity measure $\Phi$ is also rotation invariant), then (1.1) holds with equality. Other inequalities of isoperimetric type which Wieacker [36] proved include

\begin{equation}
D_0(Y) \leq \frac{\lambda_{d-1}}{d \lambda_d} D_{d-1}(X_0 Y)^d
\end{equation}

as a special case and

\begin{equation}
V_s(Y) \geq d \lambda^2_d \left( \frac{\lambda_{d-1}}{d \lambda_d} D_{d-1}(X_0 Y) \right)^{-d}.
\end{equation}

Here $D_{d-1}(X_0 Y)$ denotes the surface area density (expected surface area per unit volume) of the random set $X_0 Y$ formed by the union