On Frobenius Groups II:
Universal completion of nearfields of finite degree
over a field of reference

by

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Introduction: A group of permutations $G$ of a letter set $N$ is said to be a Frobenius group if it is transitive and if the identity permutation is the only member of $G$ fixing more than one letter.

In abstract group theory $G$ is characterized by the existence of a self normalizing subgroup $G_0$ intersecting trivially any $G$-conjugate $G_0'$. Knowing $G_0$ we obtain a faithful permutation representation of $G$ as Frobenius group by application of the left multiplicative action of $G$ on the set $N$ formed by the $G_0$-right cosets. Here $G_0$ is represented as stabilizer of the coset $G_0$, denoted also as $O$.

The elements of $G$ with no fix element, together with $1_G$, form the Frobenius kernel $K(G)$ of $G$. It is invariant under the inner automorphismus of $G$, but it is not necessarily a subgroup. On the other hand, if $N$ is finite then $K(G)$ is known to be a regular permutation group of $N$ (s.[4]) which is nilpotent in case $G_0 \ast 1$ (s.[5]).

If $K(G)$ is a regular permutation group $\ast 1$ of $N$ then we choose a second element $e$ of $N$ and define two binary operations, say $+$ and $\cdot$, on $N$ according to

\begin{align*}
(0.1a) \quad & u(O) + v(O) = uv(O) \quad (u, v \in K(G)), \\
(0.1b) \quad & a(e) \cdot x = a(x) \quad (a \in G_0, x \in N),
\end{align*}

so that $N$ turns into a $+$group $N^+$ (the additive group) and
G_0(e) turns into a group \( N^0 \) (the multiplicative group of \( N \)) with \( e \) as unit element. There holds the right distributive law

\[(0.1c) \quad a \cdot (x+y) = a \cdot x + a \cdot y \quad (a \in N^0; \ x, y \in N)\]

and the associative law of multiplication

\[(0.1d) \quad a \cdot (b \cdot x) = (a \cdot b) \cdot x \quad (a, b \in N^0; \ x \in N),\]

so that the multiplication is fix-point-free:

\[(0.1e) \quad \text{If} \quad a \cdot x = x \ast 0 \quad (a \in N^0, \ x \in N) \quad \text{then} \quad a = e.\]

The permutations belonging to \( G \) are uniquely presented in the form

\[(0.1f) \quad g = \left( \begin{array}{c} x \\ a \cdot x + b \end{array} \right) \quad (a \in N^0, \ b \in N)\]

or

\[g(x) = a \cdot x + b\]

where \( a, b \) are characterized by

\[(0.1g) \quad b = g(0) \quad a = g(e) - b,\]

and \( x \) runs through the letter set \( N \).

A set \( N \) with two binary operations \(+, \circ\) is said to be a nearfield, if \( N \) is a \(+\)group with neutral element \( 0 \), and if the elements \( a \in N \) for which \( a \cdot x \) is defined for all \( a \) of \( N \), form a group \( N^0 \), say with unit element \( e \ast 0 \), and if, moreover, the rules \((0.1c-e)\) obtain.

It follows that the permutations:

\[(0.2a) \quad \left( \begin{array}{c} x \\ a \cdot x + b \end{array} \right) \quad (a \in N^0, \ b \in N)\]

of the elements \( x \) of \( N \) form a Frobenius group \( G \) acting on \( N \). Its Frobenius kernel is the regular normal subgroup formed by the translation mappings

\[(0.2b) \quad \left( \begin{array}{c} x \\ x + b \end{array} \right) \quad (b \in N)\]

of the elements \( x \) of \( N \). The stabilizer of \( 0 \) is the subgroup \( G_0 \) formed by the left multiplication mappings.