The radii of starlikeness of certain rational functions with real simple poles

PAVEL G. TODOROV

In this paper we shall study the set $F_1$ of all normalised rational functions

$$f(z) = \sum_{k=1}^{n} \frac{A_k}{z - a_k}$$

and the set $F_2$ of their associated functions

$$\mathcal{F}(z) = f\left(\frac{1}{z}\right) = \sum_{k=1}^{n} \frac{zA_k}{1 - a_k z},$$

where

$$A_k > 0, \quad \sum_{k=1}^{n} A_k = 1, \quad -1 \leq a_k \leq a_{k+1} \leq 1, \quad 1 \leq k \leq n-1, \quad n \geq 2.$$  \(3\)

From [1], p. 416 it follows that the $(n-1)$ finite zeroes $\zeta_k (1 \leq k \leq n-1)$ of the function (1) are real, simple and separated: $a_k \leq \zeta_k \leq a_{k+1} (1 \leq k \leq n-1)$. Hence the functions (1) and (2) have the corresponding representations

$$f(z) = \frac{\prod_{k=1}^{n-1} (z - \zeta_k)}{\prod_{k=1}^{n} (z - a_k)}$$

and

$$\mathcal{F}(z) = z \frac{\prod_{k=1}^{n-1} (1 - \zeta_k z)}{\prod_{k=1}^{n} (1 - a_k z)}.$$  \(4\) \(5\)

In [1], p. 417, Theorem 1 it is proved that the open disc $|z| > 1$ is maximal domain of univalence of the class $F_1$. Hence the open disc $|z| < 1$ is maximal domain of univalence of the class $F_2$. Therefore, in the disc $|z| < 1$ the class $F_2 \subset S$, where $S$ is the well-known class of all regular and normalized univalent functions in the disc $|z| < 1$.

Here we consider the problem of finding the radii of starlikeness of the classes $F_1$ and $F_2$ with respect to $z = \infty$ and $z = 0$, respectively. (Compare with the analogous problem in [2], p. 304, Theorem 2.1 but when the points $a_k (1 \leq k \leq n, n \geq 2)$ from (3) lie in the open disc $|z| < 1$). In order to solve our
problem we adapt the method in [3], pp. 514–515, by which the radius of
starlikeness of the subset to (5) has been found when \( n = 3 \) and under the
conditions \( 0 \leq a_k \leq c_k \leq a_{k+1} \leq 1, \ k = 1, 2. \)

Now by means of the representation (5) we obtain the following result:

**THEOREM 1.** The radius of starlikeness of the class \( F_2 \) is
\[
rs = \sqrt{3(6 - \sqrt{33})} = 0.8753 \ldots ,
\]
i.e. for each function \( \mathcal{F}(z) \in F_2 \) in the disc \( |z| \leq rs \) the inequality
\[
\text{Re} \left\{ \frac{z \mathcal{F}'(z)}{\mathcal{F}(z)} \right\} \geq 0
\]
holds, where the equality holds only for the following extremal functions
\[
\mathcal{F}_0(z) = \frac{(6 + \sqrt{6 + 2\sqrt{33}})z}{12(1 + z)} + \frac{(6 - \sqrt{6 + 2\sqrt{33}})z}{12(1 - z)}
\]
and
\[
\mathcal{F}_1(z) = \frac{(6 - \sqrt{6 + 2\sqrt{33}})z}{12(1 + z)} + \frac{(6 + \sqrt{6 + 2\sqrt{33}})z}{12(1 - z)}
\]
at the two "critical points"
\[
z^*_0 = \sqrt{3(6 - \sqrt{33})} e^{\pm i\theta},
\]
and
\[
z^*_1 = \sqrt{3(6 - \sqrt{33})} e^{\pm i(\pi - \theta)},
\]
respectively, where
\[
\cos \theta_s = \frac{1}{3} \sqrt{6(\sqrt{33} - 1)}, \quad \theta_s = 27°13'16.4''.
\]

**Proof.** Via the logarithmic derivative of (5) we obtain the formula
\[
z \mathcal{F}'(z) = \frac{\sum_{k=1}^{n} \frac{1}{1 - a_k z} - \sum_{k=1}^{n-1} \frac{1}{1 - c_k z}}{\mathcal{F}(z)^n} \quad (n \geq 2).
\]
For fixed \( z = re^{i\theta}, 0 < r < 1, -\pi < \theta \leq \pi \) we set
\[
m(z) := \min_{\mathcal{F}(z) \in F_2} \text{Re} \left\{ \frac{z \mathcal{F}'(z)}{\mathcal{F}(z)} \right\} = \min_{I} \left[ \sum_{k=1}^{n} g(a_k r, \theta) - \sum_{k=1}^{n-1} g(c_k r, \theta) \right],
\]
where \( g(a_k r, \theta) \) the second "minimum" is taken over the set
\[
I = \{(a, c) \mid -1 \leq a_k \leq c_k \leq a_{k+1} \leq 1, \ 1 \leq k \leq n - 1, \ n \geq 2\}
\]