A note on biduals of strict (LF)-spaces

by

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(1) Is the bidual of a strict inductive limit of a sequence of locally convex spaces the inductive limit of the biduals?
(2) Is the bidual of a strict (LF)-space again an (LF)-space?
(3) Is the bidual of a strict (LF)-space complete?
M. Valdivia gave a (negative) answer to the first question in 1979 in [5]. Since his counterexample is not an (LF)-space, problem (2) remained open. The aim of this note is to present a negative solution to questions (2) and (3). The answer to question (2) is negative even if every step of the (LF)-space is distinguished, in which case the strong bidual is complete by a result of Grothendieck. Moreover, we show that the strong dual of a strict (LF)-space need not be countably barrelled.

0. Notations.

Given a dual pair \langle E,F \rangle of vector spaces, we denote by \sigma(E,F) resp. \beta(E,F) the corresponding weak resp. strong topology on E. If F is a locally convex space, we denote by F' its topological dual and sometimes abbreviate \( F'_b := (F', \beta(F', F)) \). By \( F'' := (F'_b)' \) we denote the bidual and write \( F'' := (F''', \beta(F''', F')) \). Whenever \( (F_n : n \in \mathbb{N}) \) is a sequence of locally convex spaces, we denote by \( \prod (F_n : n \in \mathbb{N}) \) its product equipped with the product topology and by \( \oplus (F_n : n \in \mathbb{N}) \) its direct sum provided with the topology of the locally convex direct sum. If \( U_n \subseteq F_n \) satisfies \( 0 \in U_n \) for all \( n \in \mathbb{N} \), we sometimes write \( \oplus (U_n : n \in \mathbb{N}) := \{ (x_n : n \in \mathbb{N}) \in \oplus (F_n : n \in \mathbb{N}) : x_n \in U_n \text{ for all } n \in \mathbb{N} \} \).
For the inductive limit with respect to an increasing sequence \((F_n : n \in \mathbb{N})\) of locally convex spaces with continuous inclusions \(F_n \hookrightarrow F_{n+1}\) we use the symbol \(\text{ind } F\). The inductive limit \(\text{ind } F\) is called strict if all the inclusions \(F_n \hookrightarrow F_{n+1}\) are topological isomorphism onto their ranges. If each \(F_n\) is a Fréchet space the inductive limit \(\text{ind } F\) is called an \((LF)\)-space. For the projective limit of a projective sequence \(((F_n : n \in \mathbb{N}),(q_n : n \in \mathbb{N}))\) of locally convex spaces \(F_n\) and linking continuous linear maps \(q_n : F_{n+1} \rightarrow F_n, n \in \mathbb{N}\), we use the symbol \(\text{proj } F\).

1. Preliminaries.

Let \(F\) be a locally convex space which is the union of an increasing sequence of closed linear subspaces \(F_n, n \in \mathbb{N}\), such that \(F\) carries the strongest locally convex topology which induces on each \(F_n\) the original relative topology; in particular, \(F\) is then the strict inductive limit \(\text{ind } F\).

The strong dual \(F'_b\) of \(F\) can be canonically identified with the projective limit of the projective sequence \(((F'_n, b : n \in \mathbb{N}),(q'_n : n \in \mathbb{N}))\) of the strong duals \(F'_n, b\), where \(q'_n : F'_{n+1} \rightarrow F'_n\) are the transpose to the inclusions \(F_n \hookrightarrow F_{n+1}\), and hence surjective. Indeed, the linear bijection \(F'_b \rightarrow \text{proj } F'_n, b, f \rightarrow (f|F_n : n \in \mathbb{N})\) is continuous; moreover it is open, as follows easily from the fact that every bounded subset of \(F\) is contained in some \(F_n\) (see e.g. Köthe [2: 19,4 (4), p. 223]). The natural projections \(p_n : F' \rightarrow F'_n, f \rightarrow f|F'_n\), are the transpose to the inclusions \(F_n \hookrightarrow F\) and thus they are also surjective. Consequently, for every \(n \in \mathbb{N}\), the transpose map \(p^n_t : F'' \rightarrow F''_{n}\) identifies (algebraically) \(F''\) with a subspace of \(F''\) such that \(F'' \subset F''_{n+1}\), \(n \in \mathbb{N}\), and \(F'' = \bigcup(F''_{n} : n \in \mathbb{N})\). Since all \(p^n_t : F'' \rightarrow F''_{n}\) and all \(q^n_t : F''_{n+1} \rightarrow F''_{n}\) are continuous, \(F''\) has a natural representation as the inductive limit \(F''_{\text{ind}} := \text{ind } F''_{n}, b\) and the corresponding inductive limit topology is stronger than the strong topology \(\beta(F'', F')\) on \(F''\). Thus the identity map \(F''_{\text{ind}} \rightarrow F''_{b}\) is continuous and Grothendieck's question asked whether it is also open.

Let us assume now that all the spaces \(F_n\) are quasibarrelled. In this case Grothendieck [1, p. 84/85] proved the following statements.