MORITA EQUIVALENCE AND QUOTIENT RINGS

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Let $R$ be a ring with unity. An element of $R$ is said to be regular if it is not a zero-divisor. A ring $R$ satisfies the (right) Ore condition if for every pair $a, b$ in $R$ where $b$ is regular there exist $c, d$ in $R$ with $d$ regular such that $ad = bc$, in this case we say that $R$ is a (right) Ore ring. It is well-known that $R$ is a (right) Ore ring if and only if $R$ has a (right) classical ring of quotients, that is a ring $Q$ containing $R$ in which every regular element is invertible and such that every element of $Q$ is of the form $ab^{-1}$ where $a, b \in R$ and $b$ is regular. We say that $R$ is classical if it is its own classical ring of quotients (i.e. every non zero-divisor is invertible).

Faith [7, p.229] asks if the (right) Ore condition is Morita invariant. In this direction Bergman [2, Theorem 13] constructs a classical ring whose matrix rings are not classical. However, it appears to be unknown [2, p.270] whether these matrix rings are right Ore rings.

In this note we shall prove the following:

**Theorem.** There exists a ring $R$ that is classical and such that $M_n(R)$ is neither right nor left Ore ring for every $n > 1$.

Chatters ([3],[4]) provides an example of a (right) Noetherian ring $R$ such that $R$ is (right) Ore but $M_2(R)$ is not. This shows, in particular, that the (right) Ore condition is not a Morita invariant property. It is, however, an open question [4, p.168] if there exists a (right) Noetherian ring satisfying the statements in the above theorem.

Since a classical ring is obviously an Ore ring (i.e. it satisfies the right and left Ore conditions) we see that our theorem provides an example of an Ore ring whose matrix rings are not.

**Proof of the Theorem:** It suffices to find for each $n > 1$ a classical ring $R_n$ such that $M_n(R_n)$ is not an Ore ring. (For then $R = \Pi_{n>1} R_n$ gives the desired ring).

Fix an integer $n > 1$. Let $K$ be a commutative field and let $X = (x_{ij}), Y = (y_{ij})$ be $n \times n$ matrices with indeterminate coefficients. Define $V$ as the algebra over $K$ generated by the $2n^2$ symbols $x_{ij}, y_{ij}$ with the defining relations $XY = I_n = YX$.

Every element in $V$ can be expressed as a non-commutative polynomial in the $x$'s and $y$'s with coefficients in $K$. Such an expression is said

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to be in normal form if no term has a factor $x_{ij}y_{ij}$ or a factor $y_{ij}x_{ij}$. Cohn [5] proves that any element in $V$ can be uniquely expressed in normal form, moreover the degree of the normal form is a filtration on $V$. In [5, p.p. 223-224] it is also shown that the dependence number of $V$ with respect to this filtration is $n$. Since $n > 1$, this filtration is a degree function on $V$. In particular we see that the units of $V$ belong to $K$. Consider now $V \coprod V$, the coproduct of $V$ and $V$ over $K$. $V \coprod V$ can be viewed as the $K$-algebra generated by $x_{ij}, y_{ij}, x'_{ij}, y'_{ij}$ with the defining relations $XY = YX = I_n$, and $X'Y' = Y'X' = I_n$, where $X' = (x'_{ij})$ and $Y' = (y'_{ij})$. By [6, Proposition 5.3.4] every unit of $V \coprod V$ belongs to $K$. It is also known [1, Theorem 1.6] that $V$ and so $V \coprod V$ is a fir [6, Theorem 5.3.21]. Then by Cohn's theorem there exists a field $D$ containing $V \coprod V$. On the other hand, by using normal form in the coproduct we see that the $K$-subalgebra $F = K < x_{ij}, x'_{ij} > C D$ is free on the generators $x_{ij}$ and $x'_{ij}$.

Define $R$ to be the trivial extension of the $F$-bimodule $D/V \coprod V$, that is $R = F \oplus (D/V \coprod V)$ as abelian group and the multiplication is defined by:

$$(f, \tilde{d})(f', \tilde{d}') = (ff', f\tilde{d} + \tilde{d}f').$$

First we note that $R$ is a classical ring. Suppose $(f, \tilde{d}) \in R$ is not a zero-divisor. This implies $f \neq 0$ and so $f^{-1} \in D$. But then $(f, \tilde{d})(0, f^{-1}) = 0$, so that $f^{-1} \in V \coprod V$. By the above discussion $f \in K$ and hence $(f^{-1}, -f^{-2}\tilde{d})$ is the inverse of $(f, \tilde{d})$.

We shall prove that $M_n(R)$ is not a (right) Ore ring. Consider $a = ((x_{ij}, 0))$ and $b = ((x'_{ij}, 0))$ in $M_n(R)$. We claim that $a$ and $b$ are regular elements of $M_n(R)$. For if

$$
\begin{pmatrix}
(x_{11}, 0) & \cdots & (x_{1n}, 0) \\
\vdots & & \vdots \\
(x_{n1}, 0) & \cdots & (x_{nn}, 0)
\end{pmatrix}
\begin{pmatrix}
(f_1, \tilde{d}_1) \\
\vdots \\
(f_n, \tilde{d}_n)
\end{pmatrix} = 0,
$$

we get $\sum_{i=1}^{n} x_{i1}f_i = 0$ in the free algebra $F$. So all $f_i'$s are 0. Also we have

$$
\begin{pmatrix}
x_{11} & \cdots & x_{1n} \\
\vdots & & \vdots \\
x_{n1} & \cdots & x_{nn}
\end{pmatrix}
\begin{pmatrix}
d_1 \\
\vdots \\
d_n
\end{pmatrix} =
\begin{pmatrix}
d'_1 \\
\vdots \\
d'_n
\end{pmatrix}
$$

for suitable $d'_i \in V \coprod V$. But then

$$
\begin{pmatrix}
d_1 \\
\vdots \\
d_n
\end{pmatrix} =
\begin{pmatrix}
y_{11} & \cdots & y_{1n} \\
\vdots & & \vdots \\
y_{n1} & \cdots & y_{nn}
\end{pmatrix}
\begin{pmatrix}
d'_1 \\
\vdots \\
d'_n
\end{pmatrix}.
$$