SHORT COMMUNICATION

COMMENTS ON A NOTE BY AGGARWAL

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0.

In a recent note [1], Aggarwal gave an example of a bimatrix game in which the Lemke-Howson algorithm [3], when started from all rays, gave only two of the three equilibrium points. Here we describe a complementary pivot algorithm that yields at least one and hopefully many equilibrium points of a bimatrix game. We also give an example for which the two preferable equilibrium points are inaccessible by most complementary pivot algorithms including that presented here.

1.

A bimatrix game $\Gamma(A, B)$ is specified by two $m \times n$ matrices $A, B$ where $a_{ij}$ ($b_{ij}$) is the payoff to player I (II) when player I uses pure strategy $i$ and player II pure strategy $j$. Let $e_p \in \mathbb{R}^p$ denote the vector consisting of $p$ ones. Then mixed strategy spaces for players I and II are

\[
X = \{x \in \mathbb{R}^m : x \succeq 0, x^T e_m = 1\},
\]

\[
Y = \{y \in \mathbb{R}^n : y \succeq 0, y^T e_n = 1\}.
\]

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A Nash equilibrium point for \( \Gamma(A, B) \) is a pair \((\bar{x}, \bar{y}) \in X \times Y\) such that for all \((x, y) \in X \times Y\), \(x^T A \bar{y} \leq \bar{x}^T A \bar{y}\) and \(\bar{x}^T B y \leq \bar{x}^T B \bar{y}\). It is trivially checked that the addition of any constant to all entries of \(A\) or \(B\) does not affect equilibrium points. We can thus assume without loss of generality that \(A\) and \(B\) have any prescribed sign (for each entry).

2.

For this section and the next, assume \(A > 0, B > 0\). Consider the system

\[
(*) \quad w = \begin{pmatrix} 0 & -A \\ -B^T & 0 \end{pmatrix} z + e_{m+n}, \quad w = \begin{pmatrix} s \\ t \end{pmatrix} \geq 0, \quad z = \begin{pmatrix} u \\ v \end{pmatrix} > 0.
\]

Here \(w, z \in \mathbb{R}^{m+n}, s, u \in \mathbb{R}^m\) and \(t, v \in \mathbb{R}^n\).

A solution to (*) is called complementary if \(w^T z = 0\).

The following result is analogous to Lemke and Howson's characterization [3; page 415] and Eaves' Lemma 9.28 [2; page 625].

**Theorem.** If \((\bar{x}, \bar{y})\) is an equilibrium point for \(\Gamma(A, B)\), then there is a complementary solution to (*) with \(u = \bar{x}/(\bar{x}^T B \bar{y})\) and \(v = \bar{y}/(\bar{x}^T A \bar{y})\). Conversely, if \(w, z\) is a complementary solution to (*) other than \(w = e_{m+n}, z = 0\), then \((\bar{x}, \bar{y})\) is an equilibrium point for \(\Gamma(A, B)\), where \(\bar{x} = u/(u^T e_m)\) and \(\bar{y} = v/(v^T e_n)\).

3.

A solution to (*) is called \(k\)-almost-complementary (\(k\)-a.c.) if \(w^T z = w_k z_k\), for \(1 \leq k \leq m + n\). Assume that the system (*) is nondegenerate; otherwise standard lexicographic arguments can be used, see [2]. Then the \(k\)-a.c. basic solutions form the nodes of a graph \(G_k\), with two nodes adjacent if their midpoint is also \(k\)-a.c. From the boundedness of (*), \(k\)-a.c. basic solutions have degree 1 or 2 in \(G_k\) according as they are complementary or not. Thus \(G_k\) is the union of disjoint cycles and paths linking two complementary solutions.

**Algorithm**

**Step 0.** Let the initial complementary solution be \(z = 0, w = e_{m+n}\), and let \(k = 1\).