1. INTRODUCTION

A numerical method is proposed for the fast solving of parabolic boundary control problems. The state is governed by a linear or nonlinear parabolic differential equation. If the cost function is a quadratic functional of state and control, the optimal control can be characterized by an equation of the form $u = \mathcal{K}(u)$, where the nonlinear operator $\mathcal{K}$ has some special properties mentioned in Section 3. The discretized problem (Section 4) is a large system of linear or nonlinear equations. This system can be solved by the multi-grid iteration of the second kind described in Section 5. Numerical examples are reported for a linear problem and a problem with constrained control.

2. FORMULATION OF THE CONTROL PROBLEM

Let $Q$ be the cylinder $\Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^d$, $d \geq 1$. $\Sigma_0$ is a subset of the lateral boundary $\Sigma = \partial \Omega \times (0,T)$. As a model problem we formulate the following linear problem.

Let $f, g, y_0$ be fixed functions defined on $Q$, $\Sigma - \Sigma_0$, and $\Omega$, respectively, while the control $u$ is a varying function defined on $\Sigma_0$.

$y = y(u) = y(x,t;u)$ is the solution of the following parabolic initial-boundary value problem:

(1) \[ y_t + Ay = f(x), \quad Bv = \begin{cases} u & (x,t) \in (\Sigma_0), \\ g & (x,t) \in (\Sigma - \Sigma_0) \end{cases}, \quad y(x,0;u) = y_0(x) \quad (\Omega). \]

$A$ denotes an elliptic differential operator, $B$ is a boundary operator. E.g., $A = -\Delta$, $B = \partial/\partial n$. We want to approximate a given function $z_d$ by the final state:

$y(x,T;u) \approx z_d(u)$. A possible cost function is

(2) \[ J(v) = \| y(x,T;v) - s_d \|_{L^2(\Omega)}^2 + \delta \| v \|_{L^2(\Sigma_0)}^2, \quad \delta > 0. \]

The control is varying in a convex set $U_{ad} \subset L^2(\Sigma_0)$ of admissible controls. The solution $u \in U_{ad}$ of the minimization problem

(3) \[ J(u) = \min \{ J(v) : v \in U_{ad} \} \]

can be characterized by means of the adjoint state $p(u)$, which is the solution of
the following parabolic equation with negative time orientation:

\[ -p_t + A^* p = 0 \quad (\Omega), \quad \partial p = 0 \quad (\Sigma), \quad p(x,T;u) = y(x,T;u) - z_d(x) \quad (\Omega), \]

where the adjoint operator \( A^* \) and \( C \) satisfy Green's formula

\[ (Ay, p)_{L^2(\Omega)} - (y, A^* p)_{L^2(\Omega)} = (y, \partial p)_{L^2(\Omega)} - (\partial y, p)_{L^2(\Omega)}. \]

In the case of \( A = -\Delta \), \( B = \partial / \partial n \), we have \( A^* = -\Delta \), \( C = \partial / \partial n \).

If \( U_{ad} = L^2(\Sigma_o) \) the solution \( u \) of (3) is given by

\[ u = -\frac{1}{\alpha} p(u) \big|_{\Sigma_o} \]

(cf. Lions [7]). If \( U_{ad} = \{ u \in L^2(\Sigma_o) : u \in I \text{ a.e.} \} \) with an interval \( I = [\mu_{\text{min}}, \mu_{\text{max}}] \), the optimal solution satisfies

\[ u = \left[ -\frac{1}{\alpha} p(u) \right]_{\Sigma_o} \]

where \([\xi]_I = \xi \text{ if } \xi \in I, \left[ \xi \right]_I = \mu_{\text{max}} \text{ if } \xi > \mu_{\text{max}}, \left[ \xi \right]_I = \mu_{\text{min}} \text{ if } \xi < \mu_{\text{min}}\).

The control function may also appear in other parts of Eq. (1); the Neumann condition may be replaced by a Dirichlet condition. Also the cost function can be changed (observation of the total state or observation on the boundary). The corresponding equations (4) and (6) are given in [2].

In the following we formulate a nonlinear control problem. Let \( y(u) \) be the solution of

\[ y_t + \mathcal{A}(y) = 0 \quad (\Omega), \quad \mathcal{B}(y) = \left\{ u \in L^2(\Sigma_o) : u \in I \text{ a.e.} \right\}, \quad y(x,0;u) = y_0(x) \quad (\Omega), \]

where \( \mathcal{A} \) and \( \mathcal{B} \) are nonlinear operators, e.g., \( \mathcal{A}(y) = - (a(y)y_x)_x \), \( \mathcal{B}(y) = b(y) + \partial y / \partial n \).

Denote the derivatives of \( \mathcal{A}(y) \) and \( \mathcal{B}(y) \) with respect to \( y \) at \( y(u) \) (\( u \) optimal control) by \( A \) and \( B \). Define the boundary operator \( C \) and the function \( \varphi \) by means of Green's identity

\[ (Ay, p)_{L^2(\Omega)} - (y, A^* p)_{L^2(\Omega)} = (y, \partial p)_{L^2(\Omega)} - (\partial y, p)_{L^2(\Omega)}. \]

For the example mentioned above we have

\[ A z = -(a(y)z_x)_x - (a_y(y)z_x)_x, \]
\[ A^* z = -(a(y)z_y)_x + a_y(y)y_x z_x, \]
\[ B z = b_y(y)z + \partial z / \partial n, \quad \varphi = a(y), \]
\[ \partial p = [b_y(y) - a_y(y) \partial y / \partial n] p + a(y) \partial p / \partial n. \]

The adjoint state \( p = p(y) \) is the solution of the linear parabolic equation (4).

Then the solution \( u \) of (2) can be characterized by

\[ u = -\frac{\varphi}{\alpha} p(u) \big|_{\Sigma_o} \quad \text{if } U_{ad} = L^2(\Sigma_o), \]
\[ u = \left[ -\frac{\varphi}{\alpha} p(u) \big|_{\Sigma_o} \right]_I \quad \text{if } U_{ad} = \{ v \in L^2(\Sigma_o) : v(x) \in I \text{ a.e.} \}. \]

Denoting the right-hand sides of (6a), (6b), (9a), or (9b) by \( \mathcal{K}(u) \) we obtain the equation