On a General Method for 
Solving Time-Optimal Linear 
Control Problems 
by 
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Abstract 
In this paper a class of methods for solving time-optimal linear control problems in an abstract setting is presented. Two convergent versions of this class, termed as first and second implementation of a basic algorithm, generalize the main two convergent algorithms that have been developed for linear systems governed by ordinary differential equations. 

1. Introduction. 
This article is an abbreviated version of [7] where a unified approach is given to algorithms for the computation of the minimal time and time-minimal controls for steering an abstract linear system into a time-independent target state by a family of admissible controls. 

The general algorithm, termed as basic algorithm, which is described in Subsection 3.1 is based on a duality statement (see Theorem 2.3) which characterizes the minimal time by a maximum property, if a condition is met which generalizes the concept of properness in the sense of Hermes-LaSalle [8]. This duality statement generalizes a result of Neustadt in [12], who seems to have been the first to develop an algorithm for solving time-optimal control problems. Neustadt uses the maximum property of the minimal time in order to establish a differential equation from which time-minimal controls can be computed, if the system is normal. Normality, in general, is a stronger property than properness and guarantees uniqueness of time-minimal controls. In [5] Eaton gives a procedure for solving normal time-minimal control problems with time-dependent targets which, for fixed targets, can be considered as a special case of the basic algorithm developed in Subsection 3.1. He was, however, unable to prove convergence. This is also pointed out by Boltjanski who in [3] gives a unified representation of
Neustadt's and Eaton's results. In general, it is not possible to prove convergence for the basic algorithm of Subsection 3.1. By Theorem 3.1, however, a wide class of algorithms is admitted for which convergence can be proved. Among these there are two algorithms, termed as first and second implementation which generalize the main two classes of convergent algorithms developed for linear control problems governed by ordinary differential equations in [2],[4],[9], and [6].

Due to the limited space for this publication, applications to linear control problems with ordinary or partial differential equations cannot be presented. The interested reader is referred to [7].

2. Controllability and Time-Minimal Controllability.
We consider the following abstract version of a linear control problem: Let $X$ be the dual space $Z^*$ of a separable Banach space $Z$, let $\{S_t\mid t\in[0,T]\}$, for some $T>0$, be a family of continuous linear mappings from $X$ into a finite-dimensional normed linear space $Y$ such that $S_0$ maps $X$ into the origin of $Y$, and let $\hat{y}\in Y$ be a fixed element with $\hat{y}\neq 0$. Further let, for some constant $M>0$,

$$U_M = \{u\in X|\|u\|\leq M\}. \quad (2.1)$$

Each element $u\in X$ is considered as a control of a physical system whose states are given by the elements of $Y$. The development of the system under a fixed control $u\in X$ with respect to the time is assumed to be described by the mapping $t\mapsto S_t(u), t\in[0;T]$. The controls which lie in $U_M$ (2.1) are called admissible. The state $\hat{y}\in Y$ is considered as a desired target state.

The problem of controllability then reads as follows:
Does there exist, for a given time $t\in(0,T]$, an admissible control $u$ such that

$$S_t(u) = \hat{y}, \quad (2.2)$$

i.e., is it possible to reach the target state $\hat{y}$ by an admissible control within the time interval $[0,t]$? A necessary and sufficient condition for controllability which was derived by Antosiewicz in [1] for linear systems governed by ordinary differential equations (see also [11]) is the content of

Theorem 2.1: For each $t\in(0,T]$ we assume the mapping $S_t:X\to Y$ to be continuous with respect to the weak* convergence in $X$. Then, for some $t\in(0,T]$ there exists an admissible control $u$ with (2.2) if, and only if

$$y^*(\hat{y}) \leq M S_t^*(y^*) \quad \text{for all } y^*\in Y^* \quad (2.3)$$