ON $L^2$ AND NON-$L^2$ MULTIPLE STOCHASTIC INTEGRATION

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0. Introduction. In their pioneering works K. Ito [4], [5] and Wiener [17] developed the theory of multiple stochastic integrals (m.s.i.)

$\int_T^n f(t_1,...,t_n)Z(dt_1)...Z(dt_n)$

where $f \in L^2(T^n)$ is a deterministic function on $T^n = T \times ... \times T$ (n times) (usually $T = \mathbb{R}^d$, $d \geq 1$), and $Z = Z(dt)$ is Gaussian or Poisson noise in T. (By noise we mean a finitely additive random set function with independent values on nonintersecting sets). A fundamental feature of the classical Ito-Wiener m.s.i. is their orthogonality and completeness: any random variable (r.v.) $\xi \in L^2(\Omega)$ can be uniquely expanded in orthogonal series of m.s.i., convergent in $L^2(\Omega)$.

There are however other motives to discuss m.s.i. apart from orthogonal expansions. One of them is the possibility to construct by them probabilistic objects with certain desirable properties as in the case of self-similar random fields [2], [14], [15].

Therefore it seems natural to consider also

(a) m.s.i. with respect to more general $L^2$-noises (i.e. noises with finite variance) that Gaussian or Poisson, which are $L^2(\Omega)$-r.v., or

(b) m.s.i. with respect to $L^2$-noises which are not $L^2(\Omega)$-r.v. or m.s.i. with respect to non-$L^2$-noises (among which stable noises are most important).

We shall refer to (a) as $L^2$-m.s.i. and to (b) as non-$L^2$ m.s.i. Generalization of the classical definition of m.s.i. in case (a) is straightforward, the only new problem arising is the completeness of the system of such m.s.i. Moreover, m.s.i. of type (a) can be reduced to classical m.s.i. with respect to Gaussian and Poisson noise in $T \times \mathbb{R}$ [5] as Z itself can be represented in such a way. Nevertheless we regard the question of completeness of m.s.i. in case (a) worth discussion, especially as a clear answer to it in terms of characteristics of Z can be obtained (see below).
Let us briefly review the content of the remaining sections. In Sect. 1 we define in the usual way (via integral sums with discarded diagonals) m.s.i.

\[(0.2) I^{(n)}(f) = \int_{T^n} f_{i_1}^{(t_1)} \ldots f_{i_n}^{(t_n)} Z^{(t_1)} \ldots Z^{(t_n)} dt_1 \ldots dt_n \]

with respect to \(\mathbb{R}^m\)-valued \(L^2\)-noise \(Z = (Z_1, \ldots, Z_m)\) in \(T\) with characteristic function (ch.f.)

\[(0.3) \quad \mathbb{E}[\exp\{i(a,Z(A))\}] = \exp\{ \int w(t,a) \mu(dt) \}, \quad a \in \mathbb{R}^m,\]

where

\[(0.4) \quad w(t,a) = -1/2 \sigma(t)a,a + \int_{\mathbb{R}^m} (e^{i(a''u') - l_i(a,u)}) \pi(t,du)\]

under general conditions on the 'reference measure' \(\mu(dt)\), 'diffusion matrix' \(\sigma(t) = (\sigma_{ij}(t))_{i,j=1,m}\) and the 'Lévy measure' \(\pi(t,du)\). Integral \((0.2)\) is well-defined for any (measurable) function \(f = f^{(t_1, \ldots, t_n)} \in \mathbb{L}^{1,m}\) such that

\[(0.5) \quad ||f||_n = (\int_{T^n} f_{i_1}^{(t_1)} \ldots f_{i_n}^{(t_n)} \sum_{j_1}^{(t_1)} \ldots \sum_{j_n}^{(t_n)} r_{i_1,j_1}(t_1) \ldots r_{i_n,j_n}(t_n) d\mu^n)^{1/2} < \infty\]

where \(r_{ij}(t) = E[Z_i(dt)Z_j(dt)]/\mu(dt), i,j=1,\ldots,m\) is the covariance (density) matrix of \(Z\), denoted by \(R(t) = (r_{ij}(t))_{i,j=1,m}\). The basic properties of m.s.i. \((0.2)\) will be discussed.

In Sect. 2 we investigate the completeness of m.s.i. system \((0.2)\) in \(L^2(\Omega) = L^2(\Omega,F,P)\), where \(F\) is the \(\sigma\)-algebra generated by \(Z\). It turns out that this system is complete iff the equality

\[(0.6) \quad 2\text{Re}w(t,a) + (R^{-1}(t)\text{grad}w(t,a),\text{grad}w(t,a)) = 0, \quad a \in \mathbb{R}^m\]

holds \(d\mu\)-a.e. in \(T\) (Th.2.4) (grad in \((0.6)\) refers to differentiation with respect to \(a \in \mathbb{R}^m\)). In any case, the left hand side of \((0.6)\) is negative (\(\leq 0\)). This inequality implies another interesting one:

\[(0.7) \quad |\varphi(a)|^2 + (S^{-1}\text{grad}\varphi(a),\text{grad}\varphi(a)) \leq 1, \quad a \in \mathbb{R}^n.\]