1. Introduction

Many methods have been proposed for the numerical solution of deterministic optimal control problems (cf. Bryson and Ho, 1969). Early methods attempted, with limited success, to solve the two-point boundary value problem arising from Pontryagin's necessary conditions for an optimum by a shooting method, and the more recent "multiple shooting method" of Bulirsch et al. (1977) is designed to deal with the inherent instability in this technique. Miele and his co-workers (Miele, 1973; Heidemann and Levy, 1975) developed the "sequential gradient-restoration algorithm" which by-passes the stability problems by solving a sequence of linear two-point boundary value problems, using an extension of the gradient or conjugate-gradient method to a function-space. For the unconstrained case, extension of the variable-metric method (Tokumaru et al., 1970) to function-spaces has been made.

An alternative approach is to use a finite-dimensional representation of the control, and hence reduce the problem to a finite-dimensional optimization problem. Piecewise constant control functions were used for unconstrained problems by Horwitz and Sarachik (1968), and for problems without path constraints by Pollard and Sargent (1970). However, the latter authors used a simple projection technique to deal with control constraints and a penalty function to deal with terminal state constraints. In this paper we use a finite-dimensional representation of the control to formulate the general optimal control problem as a nonlinear programme, which can then be solved by a standard algorithm. Objective and constraint functions are evaluated by forward integration of the system equations, while their gradients with respect to the decision variables are obtained via backward integration of an adjoint system; the special structure of these equations makes possible the efficient implementation of integration procedures suitable for stiff systems, yielding an effective general-purpose programme.

2. Optimal Control Problem Formulation

The system is described by a set of state variables \( x(t) \in \mathbb{R}^n \), which evolve under the influence of controls \( u(t) \in \mathbb{R}^m \) according to the equation

\[
\dot{x}(t) = f(t, x(t), u(t), v), \quad t \in \left[ t_o, t_f \right]
\]

(1)

where \( [t_o, t_f] \) is the time interval of interest, and \( v \in \mathbb{R}^r \) is a vector of design parameters for the system which satisfy constraints.
The class of admissible controls \( u(t) \) consists of piecewise-continuous functions of \( t \) which satisfy

\[
a^u \leq u(t) \leq b^u, \quad t \in [t_0, t_f].
\]

The initial state \( x(t_0) \) satisfies the conditions

\[
a^o \leq x(t_0) \leq b^o.
\]

For almost all \( t \in [t_0, t_f] \), and for all possible values of \( v, x(t) \) and \( u(t) \), the vector-valued function \( f(t, x, u, v) \) satisfies the conditions:

(i) It is a piecewise-continuous function of \( t \) and a differentiable function of \( x, u \) and \( v \).

(ii) There exists a function \( S(t) \) summable on \([t_0, t_f]\), and a function \( \psi(z) \) positive and continuous for \( z \geq 0 \) but not summable on \( [0, \infty) \), such that

\[
|f(t, x, u, v)| \leq S(t)/\psi(|x|).
\]

(iii) The derivatives \( f_u(t, x, u, v), f_v(t, x, u, v) \) and \( f_x(t, x, u, v) \) are bounded and \( f_x(t, x, u, v) \) is Lipschitz continuous in \( x \).

These conditions ensure that for each admissible set of design parameters, initial state and control there is a unique solution \( x(t) \), \( t \in [t_0, t_f] \), to equation (1), and further that these solutions are absolutely continuous in \( t \) and uniformly bounded over all admissible choices of design parameters, initial states and controls.

The state and controls may also be subject to path and terminal constraints of the form

\[
\begin{align*}
a^g(t) \leq g(t, x(t), u(t), v) & \leq b^g(t), \quad t \in [t_0, t_f] \quad (5) \\
a^c \leq F(t_f, x(t_f), v) & \leq b^c
\end{align*}
\]

where \( g(t, x, u, v) \in \mathbb{R}^p \) and \( F(t, x, v) \in \mathbb{R}^q \).

System performance is measured in terms of the scalar objective function

\[
J = F_o(t_f, x(t_f), v) \quad (7)
\]

Henceforth \( F_o(t, x, v) \) will be taken as the first element of the \((q+1)\)-dimensional vector \( F(t, x, v) \), which is assumed to be continuously differentiable with respect to its arguments, with the derivative \( F_x(t, x, v) \) Lipschitz continuous in \( x \). The function \( g(t, x, u, v) \) satisfies the conditions (i), (ii) and (iii) given above.

The optimal control problem is to choose an admissible set of design parameters, initial state and controls, and possibly also the final time, to minimize \( J \), subject to the conditions (1)-(6).