Homotopy theoretic control problems are for the first time introduced in nonlinear differential geometric control theory such as: homotopic controls, $\mathcal{E}$-controlled homotopic invariant distributions of vector fields and almost decoupling of nonlinear systems; (feedback) homotopic equivalence of nonlinear control systems; nonlinear systems that generate control homotopies, etc.

1. INTRODUCTION

Elements of homotopy theory were used in several control problems of the recently developed nonlinear differential geometric control theory.

Brockett and Wesley Wilson Jr. have used homotopy spheres and homotopies in the topologic study of the level surfaces of the Lyapunov functions $[1,20]$. Brockett raised an interesting problem in $[2]$ concerning the topological properties of the reachable set of bilinear systems: how is the homotopy type of the reachable set changing dynamically in time.

Byrnes, Isidori, Krener and others obtained homotopy type obstructions to global $(f,g)$-invariance $[5,12,14]$.

Elliott, Sussmann, Jurdievic and Hermes proved some results on the influence of the homotopy groups of manifolds over the controllability of the nonlinear control systems $[9,11,17]$.

Some original homotopy theoretic problems are for the first time formulated and discussed in this paper as control problems in the nonlinear differential geometry control theory: homotopic controls, controlled homotopic invariant distributions of vector fields and almost decoupling of nonlinear systems; feedback homotopic equivalence of nonlinear control systems; nonlinear systems that generate control homotopies and others. Some new types of homotopies are defined to serve as mathematical instruments in control theoretic pr -
blems: limit homotopies, control homotopies (different from control-
led homotopies).

This paper is a continuation of the author's work and interest in approximations, perturbations and deformations in nonlinear diffe-
rential geometric control \([4,5]\). The present work outlines the full program carried in \([3]\) by the author. Another element of (algebraic) topology, homology groups, was used by the author and D. Iva~cu to study the controllability of dynamical systems\([13]\).

2. MATHEMATICAL PRELIMINARIES

In this section, some elements of homotopy theory are intro-
duced following \([16]\) that will be used in the sequel. The classical textbook of Whitehead is recommended as a reference work on homotopy theory \([21]\).

Let \(X\) and \(Y\) be two topological spaces. We denote by \(\text{Top}(X, Y)\) the set of all continuous maps \(\tilde{f}: X \to Y\). Two maps \(\tilde{f}_0, \tilde{f}_1 \in \text{Top}(X, Y)\) are said to be homotopic (we write \(\tilde{f}_0 \sim \tilde{f}_1\)) if there exists a con-
tinuous map \(F: X \times [0,1] \to Y\) called homotopy of \(\tilde{f}_0\) with \(\tilde{f}_1\) such that \(F(x,0) = \tilde{f}_0(x), F(x,1) = \tilde{f}_1(x)\) for all \(x \in X\).

Example. Let \(X\) be an arbitrary topological space and \(Y\) a con-
 vex subset of \(\mathbb{R}^n\), and arbitrary \(\tilde{f}_0, \tilde{f}_1 \in \text{Top}(X, Y)\). The map \(F: X \times [0,1] \to Y\) defined by \(F(x,t) = (1-t)\tilde{f}_0(x) + t\tilde{f}_1(x)\) (where \([0,1]\)) is a homotopy of \(\tilde{f}_0\) with \(\tilde{f}_1\).

Two topological spaces, \(X\) and \(Y\) are homotopically equivalent if there exists a continuous map \(\tilde{f}: X \to Y\) called homotopical equiva-
 lence such that there exists \(g: Y \to X\) satisfying \(g \circ \tilde{f} \sim 1_X\) and \(\tilde{f} \circ g \sim 1_Y\). We used the usual notation for the homotopic maps introduced before. The homotopical equivalence is an equivalence relation.

Let \(\mathcal{U} = \{U_\lambda | \lambda \in \Lambda\}\) be a cover of a space \(Y\). Two maps \(\tilde{f}, \tilde{g}: X \to Y\) are \(\mathcal{U}\)-close if for every \(x\) in \(X\) there exists \(\lambda\) in \(\Lambda\) such that \(\tilde{f}(x), \tilde{g}(x) \in U_\lambda\). A homotopy \(H: X \times [0,1] \to Y\) is a \(\mathcal{U}\)-homotopy provided that for all \(x\) in \(X\) there exists a \(\lambda\) in \(\Lambda\) such that \(H(x,1) \in U_\lambda\). A particular interesting for us type of \(\mathcal{U}\)-homotopy is the \(\mathcal{E}\)-control-
led homotopy also termed \(\rho^{-1}(\mathcal{E})\)-homotopy \([7]\). We say that a map \(\tilde{f}: X \to Y\) is a \(\rho^{-1}(\mathcal{E})\)-homotopy equivalence provided that there exists a map \(g: Y \to X\) for which there are homotopies \(\tilde{f} \circ g \sim 1_X\) and \(g \circ \tilde{f} \sim 1_Y\) that are \(\mathcal{E}\)-controlled as follows: the homotopy \(\tilde{f} \circ g \sim 1_X\) is a \(\rho^{-1}(\mathcal{E})\)-homotopy in \(Y\) and the composition of the homoto-
py \(g \circ \tilde{f} \sim 1_Y\) with \(\tilde{f}\) gives a \(\rho^{-1}(\mathcal{E})\)-homotopy in \(Y\). Here \(\rho: Y \to B\) is the controlling map to a metric space \(B\) (the parameter space).

A \(\rho^{-1}(\mathcal{E})\)-homotopy in \(Y\) is a homotopy for which the \(\rho\)-image of the