SELF-ENTRAINMENT OF A POPULATION OF COUPLED NON-LINEAR OSCILLATORS

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Temporal organization of matter is a widespread phenomenon over a macroscopic world in far from thermodynamic equilibrium. A previous study on chemical instability[1)] implies that a simplest nontrivial model for a temporally organized system may be represented by a macroscopic self-sustained oscillator Q obeying the equation of motion

$$\dot{Q} = (i\omega + \alpha)Q - \beta|Q|^2 Q \,, \qquad \alpha,\beta > 0. \tag{1}$$

Consider a population of such oscillators Q_1, $Q_2, \cdots Q_N$ with various frequencies, and introduce interactions between every pair as follows.

$$\dot{Q}_s = (i\omega_s + \alpha)Q_s + \sum_{r \neq s} v_{rs}Q_r - \beta|Q_s|^2 Q_s \,, \qquad r,s = 1,\, 2, \cdots N \,. \tag{2}$$

We found that it is possible to construct from (2) a soluble model for a community exhibiting mutual synchronization or self-entrainment above a certain threshold value of the coupling strength. Such a type of phase transition has been considered by Winfree[2)] without resorting to specialized models but only phenomenologically.

Our simplifying assumptions are:

(I) $v_{rs} = v/N$ independently of r and s,
(II) $\alpha,\beta \to \infty$ but α/β, ω_s, v = finite,
(III) $N \to \infty$.

Let us put $Q_s = \rho_s e^{i\varphi_s}$. Owing to the assumption (II), the amplitude ρ_s may be fixed at $\sqrt{\alpha/\beta}$. Thus we have only to consider the equation

$$\dot{\varphi}_s = \omega_s + \frac{v}{N}\sum_r \sin(\varphi_r - \varphi_s) \,. \tag{3}$$

As an illustration, we summarize the results obtained when the distribution of the native frequency is a Lorentzian with the peak at ω_0 and the width γ. In this case the threshold condition is

$$\eta \equiv 2|\gamma/v| = 1 \,. \tag{4}$$

For $\eta<1$ the asymptotic behavior of $\varphi_s(t)$ as $t\to\infty$ has two possibilities depending on the value of its native frequency ω_s. Thus we classify the oscillators into two groups :

(A) $\left|\frac{\omega_s - \omega_0}{v\sqrt{1-\eta}}\right| < 1$.

The oscillators satisfying this condition are mutually synchronized and one finds

$$\varphi_s(t) = \tilde{\omega}_s t + \psi_s + \psi' , \tag{5}$$

$$\tilde{\omega}_s = \omega_0 , \tag{6}$$

where ψ_s is a single-valued function of ω_s and ψ' is an arbitrary constant but should be the same for all s belonging to this group.

(B) $\left|\frac{\omega_s - \omega_0}{v\sqrt{1-\eta}}\right| > 1$.

The oscillators of this group fail to synchronize, but their frequencies are shifted. We find

$$\varphi_s(t) = \tilde{\omega}_s t + f_s(t), \tag{7}$$

where

$$\tilde{\omega}_s = \omega_0 + (\omega_s - \omega_0)\sqrt{1 - \frac{v^2(1-\eta)}{(\omega_s - \omega_0)^2}} \tag{8}$$

and

$$f_s(t) = f_s(t + \frac{2\pi}{\tilde{\omega}_s}) . \tag{9}$$

If we define an order parameter σ by the number of the oscillators belonging to (A) divided by N, then we find

$$\sigma = \begin{cases} \frac{2}{\pi}\,\mathrm{Tan}^{-1}\,\frac{2\sqrt{1-\eta}}{\eta} & (\eta<1) \\ 0 & (\eta>1) . \end{cases} \tag{10}$$

Finally the stationary distribution of the effective frequency ω may be expressed as

$$f(\tilde{\omega}) = \sigma\delta(\tilde{\omega} - \omega_0) + \frac{\gamma}{\pi}\,|\tilde{\omega} - \omega_0|/[\{(\tilde{\omega} - \omega_0)^2+\gamma^2+v^2\chi\}\sqrt{(\tilde{\omega} - \omega_0)^2+v^2\chi}] \tag{11}$$

where

$$\chi = \begin{cases} 1-\eta & (\eta<1) \\ 0 & (\eta>1). \end{cases}$$

An infinitely sharp peak at $\tilde{\omega}=\omega_0$ corresponds to a macroscopic oscillation. The background represents the contribution from the oscillators which have failed to synchronize with the main oscillation. The intensity of the background near the center has been reduced drastically due to the factor $|\tilde{\omega}-\omega_0|$. It is very interesting to notice