Stone Duality for Stable Functions

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Introduction

The problem of finding an algebraic structure for stable open subsets of a suitable domain has been recently raised by several authors. Specifically, it could be useful to have, in the stable case, a notion similar to the one of "frame", in order to develop something similar to pointless topology, that is an algebraic insight of spaces.

In domain theory, an interesting application of frame theory is the logic of domains developed by Abramsky (see [Ab]) in the topological case.

The idea of constructing a logic for "stable properties" has been originated by Zhang (see [Zha]), who got very interesting results about "stable opens" of dI domains. However, his notions remain concrete, and it is not clear whether they give rise to a duality. In that sense, they lack the "localic" properties which justify the canonicity of Abramsky's approach.

We introduce the notion of S-structure as the structure of the algebra of stable open sets intended to correspond to the concept of frame in the stable case. These S-structures have properties which are very similar to the ones of frames, from the point of view of duality. So we may hope to achieve a logic of domains as natural as Abramsky's one, but expressing properties of programs which are not captured by the continuous approach.

In this paper, we give the fundamentals of S-structures theory. We prove first general duality results which do not involve any domain theoretical assumption about spaces. Actually we introduce the S-spaces which play wrt S-structures the same role as topological spaces wrt frames. These duality results can be specialized to the case of domains, and then we obtain a result similar to known Scott-topological duality in domain theory. The corresponding notion of domain widely subsumes the usual dI domains to which stability theory is usually restricted. Indeed these domains are the most general ones where stability makes sense. By duality method it is rather simple to treat function spaces and we retrieve cartesian closedness of the category of L-domains and stable functions which has been recently proved by P. Taylor.

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1 Basic definitions and results

This section is intended to provide an abstract definition of the algebraic structure of stable open sets. Let us first give some intuition for the forthcoming definitions. Let us consider a dI domain $D$. We call stable open subset of $D$ the inverse image of $T$ under a stable map from $D$ to the two points domain $O = \{ \bot, \top \}$. This is a natural definition since a Scott open subset is the inverse image of $T$ under a continuous function. The study of stable open sets has been initiated by Zang [Zha] and in this sense we follow his approach. As Zhang remarked stable opens are closed under finite intersection and disjoint union. Moreover they are closed under union of directed families, but not under union of arbitrary sets. In order to keep distributivity these kinds of join are the only ones we will consider.

Definition 1 In a meet-semilattice with $0$, we say that a subset $D$ is disjoint-directed (dd for short) if for any $u, v \in D$, if $u \wedge v \neq 0$ then there exists a $w \in D$ such that $u \leq w$ and $v \leq w$.

Lemma 1 If $D$ is dd not containing $0$, then the binary relation $\sim_D$ defined by: $u \sim_D v$ iff there exists $w \in D$ such that $u, v \leq w$ is an equivalence relation the classes of which are directed.

Proof: We have just to check transitivity of $\sim_D$; if $u \sim_D v$ and $v \sim_D u'$ then we have $w, w' \in D$ which verify $u, v \leq w$ and $v, u' \leq w'$; it follows that $0 \neq v \leq w \wedge w'$ so there exists $w''$ such that $u, u' \leq w''$.

Definition 2 A S-space $X$ is a pair $(X, \Omega_S(X))$ where $X$ is a set (the set of points) and $\Omega_S(X)$ is a subset of $\mathcal{P}(X)$ containing $\emptyset$, $X$ and which is closed under finite intersections and dd unions ($\Omega_S(X)$ is the set of S-open subsets of $X$, a S-topology on $X$).

If $X$, $Y$ are S-spaces, a function $f : X \to Y$ is a S-map if it preserves S-opens under inverse image.

The category of S-spaces and S-maps is noted $\text{Ssp}$.

Definition 3 A meet-semilattice $(\mathcal{L}, \wedge, 0)$ is an S-structure if

i) any dd subset $D$ of $\mathcal{L}$ has a lub noted $\vee D$

ii) finite glbs distribute over dd lbs

iii) there is a top element noted 1.

If $\mathcal{L}$ and $\mathcal{M}$ are two S-structures, a function $f : \mathcal{L} \to \mathcal{M}$ is an S-morphism if it preserves the structure.

The category of S-structures and S-morphisms is noted $\text{Sstr}$.

Definition 4 An element $u \neq 0$ of an S-structure $\mathcal{L}$ is

- connected if for any $v, w \in \mathcal{L}$ such that $v \wedge w = 0$, if $v \vee w \geq u$ then $v \geq u$ or $w \geq u$