The word problem on a monoid admits two natural generalizations:
- the first one is the extension from monoids to categories. In this case the words become the "paths" and the equality problems take the form of diagram commutation problems.
- the second is the extension from monoids to universal algebras. In this case, the words become the "terms", and the rewriting systems set the rules for their equality.

Is it possible to unify these two extensions? In this paper we answer as follows: the rewriting problem for terms is nothing but a 2-dimensional path problem in a 2-category. This observation leads to the general problem for n-paths in an n-category, or even in an ω-category. A lot of computations made by categoricians are 1-, 2- or 3-dimensional computations, and in fact n-dimensional computations take place in an (n+1)-category (see §1.2). Furthermore, beyond the unity so given to various word problems, the link with combinatorial topology appears, word problems being in this setting refinement of homotopy theory (see §1.3).

1. General setting

We are going to define the n-dimensional word problem (n∈ℕ), and more generally, a word problem of variable dimension which means computing in an ω-category (such a computation being often a reduction to a canonical form, but more generally being the construction of a "homotopy" between two expressions).

1.1. ω-Categories.

One will find in [Bu2] more synthetical definitions of ω-categories, but the definition presented below fits pretty well our purpose. Finally the whole construction amounts to the juxtaposition of an infinity of 2-categories.

An ω-graph \( G \) is the datum of a diagram of sets

\[
(*) \quad G_0 \leftarrow G_1 \leftarrow G_2 \leftarrow \ldots \leftarrow G_n \leftarrow G_{n+1} \leftarrow \ldots
\]

such that, for every \( n \in \mathbb{N} \), the following equations hold:

\[
a_n a_{n+1} = a_n b_{n+1}, \quad b_n a_n = b_n b_{n+1}.
\]
The elements of $G_n$ are named $n$-cells, and the following representations of 0-, 1- and 2-cells are well-known:

$$
\begin{align*}
X & \xrightarrow{f} Y \\
(n=0) & \\
\frac{g}{\lambda} & \xrightarrow{g \circ f} Y \\
(n=1) & \\
\frac{g}{\lambda} & \xrightarrow{g \circ f} Y \\
(n=2) & 
\end{align*}
$$

where $X,Y \in G_0$, $f,g \in G_1$, $\lambda \in G_2$.

Symbols are sometimes omitted, as for instance here:

$$
\xrightarrow{f \lambda}.
$$

We also need higher dimensional cells, and representing them is possible (although difficult), as for instance, for a 3-cell:

$$
\xrightarrow{f \lambda \mu}.
$$

In fact, as we shall see, it seems that the main part of the calculus always works in 2-graphs.

A $n$-graph is an $\omega$-graph $G$ such that $G_p = \emptyset$ for every $p>n$, and such an $n$-graph will be identified with the diagram (*) truncated at the level $n$. In particular a 0-graph is just a set, and a 1-graph is an (oriented) graph in the usual sense.

Starting from an $\omega$-graph $G$, we define, for every $0 \leq i < j$, a graph $G_{ij}$:

$$
\begin{align*}
G_i & \xleftarrow{a_{i,j}} G_j \\
& \xleftarrow{b_{i,j}} G_j
\end{align*}
$$

with:

$$
a_{i,j} = a_{1} \cdots a_{i-1} \quad \text{and} \quad b_{i,j} = b_{1} \cdots b_{j-1}.
$$

and we define, for every $0 \leq i < j < k$, a 2-graph $G_{ijk}$:

$$
\begin{align*}
G_i & \xleftarrow{a_{i,j}} G_j \\
& \xleftarrow{b_{i,j}} G_j
\end{align*}
$$

In order to obtain an $\omega$-category on $G$, we need only a category structure on each $G_{ij}$ in such a way that the 2-graphs $G_{ijk}$ become 2-categories.

It remains to define a 2-category on a 2-graph $G_0 \xleftarrow{a_0} G_1 \xleftarrow{a_1} G_2$.

essentially such a category needs the additional data of two "unity" maps:

$$
X \xrightarrow{f \cdot \text{id}(X)} X, \quad \text{and} \quad X \xrightarrow{f} Y \xrightarrow{g \circ f} Y,
$$

and of three "composition laws":

$$
\begin{align*}
X & \xrightarrow{f} Y \xrightarrow{g} Z \\
& \xrightarrow{g \circ f} Z
\end{align*}
$$

$$
\begin{align*}
X & \xrightarrow{f \cdot \lambda} Y \xrightarrow{g \cdot \mu} Z \\
& \xrightarrow{g \cdot \mu} Z
\end{align*}
$$

These data must satisfy the neutrality and associativity axioms, and the "Godement rule".