POWERSDOMAINS AS ALGEBRAIC LATTICES
PRELIMINARY REPORT
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1. Introduction

Mathematical (denotational) semantics of programming languages has been given firm theoretical foundations by the work of Dana Scott extending over the last fifteen years. He has shown that data types can be viewed as objects with a lattice-like structure (called domains), and computations on such data give rise to continuous functions on domains. Scott's theory has been developed in a series of papers (e.g. SCOTT [5], [6], [7], [8]), and is quite satisfactory for description of sequential, deterministic computations. However, the denotational approach has met considerable theoretical difficulties in approaching nondeterminism, parallelism, and concurrency. In the first place, these features often give rise to noncontinuous, or even nonmonotone, functions. Secondly, even the important case of "continuous", bounded nondeterminism presents problems. PLOTKIN [4] and SMYTH [9] invented powerdomains to represent this kind of nondeterminism in denotational semantics. Smyth powerdomains, based on Smyth ordering, fit well into Scott's theoretical framework, but provide only a rather coarse description. Plotkin uses much finer and more natural Egli-Milner ordering, but Plotkin powerdomains are far from being lattices; i.e., they are not domains in the sense of Scott. Plotkin is thus forced to extend the category of domains. The resulting SFP-objects are rather complicated to define and hard to work with; that is probably the reason why they have not been much used.

Plotkin seems to have considered the possibility of completing SFP-objects into algebraic lattices, but dismissed it as impractical. On p. 463 in [4] he writes: "So converting \( P[D] \) into a lattice would require one to add many points - not just a top element. It is not clear to the author how to keep these separate from the bona-fide elements." And again on p. 482: "...we can embed any SFP-object in \( \Phi(\omega) \)... and this gives rise to a lattice with intermediate points. But these intermediate points seem to clutter up the domain...what is wanted is a simple development of \( P[.] \) in the context of \( \Phi(\omega) \) or a similar simple structure. In Scott's words, we want an analytic, not a synthetic, development." The purpose of this work
is to do just that. It appears that one can make powerdomains based on Egli-Milner ordering into algebraic lattices. The construction appears to behave quite nicely both in its mathematical properties, and from the point of view of applications to description of non-determinism. Plotkin powerdomain is embedded as a cofinal subset into this construct, elements of Plotkin powerdomain can be easily distinguished from the "new" elements, and the "new" elements can be given a meaningful intuitive interpretation.

The rest of the paper outlines our approach to powerdomain construction. Many details and all proofs have been left out; they will appear in the full version.

I first learned about Scott's theory in the course of the International Summer School on Theoretical Foundations of Programming Methodology in Marktoberdorf in Summer 1981. I am grateful to the organizers and the NATO Science Committee for supporting my participation.

2. Domains and powerdomains

A poset \((D, \sqsubseteq)\) has the bounded join property (bj) if every finite subset \(X \subseteq D\) of \(D\) that has an upper bound in \(D\), has a least upper bound \(\text{lub}(X)\). In particular, \(\bot = \text{lub}(\emptyset)\) is the least element of \(D\). A poset \((D, \sqsubseteq)\) is complete if \(\text{lub}(X)\) exists for every directed subset \(X \subseteq D\).

Let \((D, \sqsubseteq)\) be a complete poset with the bj-property. \(d \in D\) is a compact element if for every directed \(X \subseteq D\), if \(d \sqsubseteq \text{lub}(X)\) then \(d \sqsubseteq x\) for some \(x \in X\). The poset \((D, \sqsubseteq)\) is algebraic if \(d = \text{lub}\{x \in D \mid x \sqsubseteq d, x\text{ compact}\}\) holds for all \(d \in D\).

A domain is a complete algebraic poset with the bounded join property.

This definition is equivalent to that of SCOTT [8]. It means that domains are just a slight generalization of algebraic lattices: domains are algebraic lattices without the top element. Since for the purposes of denotational semantics it is more convenient to dispense with the top element, we work with domains, although all our results remain true in the category of algebraic lattices as well.

Let \((D, \sqsubseteq)\) be a poset with the bj-property. \(I \subseteq D\) is an ideal if every finite \(F \subseteq I\) has an upper bound in \(D\) and \(x \sqsubseteq \text{lub}(F)\) implies \(x \in I\). For any \(d \in D\), \((d) = \{x \in D \mid x \sqsubseteq d\}\) is the principal ideal generated