Indecomposable representations of some Graded Lie algebras like the algebra of parabose oscillators and \( G_{2}(2) \) are studied.

1. Introduction

The representation theory of graded Lie algebras (also called Lie superalgebras) has been widely used in supersymmetry. A complete classification of simple Lie superalgebras has already been given [1]. Many results on finite dimensional representations have been derived, but relatively few on infinite dimensional representations have been worked out. We present here two such studies, one on the algebra of parabose oscillators [2] and the \( G_{2}(2), \mathbb{Z}_{2} \)-graded Lie algebra of \( sl(2) \) [3]. We explicitly obtain the indecomposable (not fully reducible) representations in these cases. The merit of this study lies in the fact that the usual irreducible representations are intimately tied up with these indecomposable representations in the sense that they are either induced on quotient spaces or subduced on invariant subspaces of these indecomposable representations.
2. Parabose oscillators

The algebra of parabose oscillators is a prototype version of a graded Lie algebra. It is defined [4] as by

\[ [a, N] = a, \]
\[ [N, a^+] = a^+, \]

and

\[ N = \frac{1}{2} \{a, a^+\} = \frac{1}{2} (aa^+ + a^+a) \]

(2.1)

where the operator \( N \) has a nonnegative infinite dimensional spectrum. The creation operator \( a^+ \) and the annihilation operator \( a \) do not satisfy the commutation relation of the usual harmonic oscillator. The normal Fock representation has been obtained earlier [5] by recognizing that \( \frac{1}{2}a^2 + \frac{1}{2}a^2 \) and \( \frac{1}{2}N \) close under commutation and the algebra is that of the Lorentz group \( S_0(2,1) \). By using the standard representation of this algebra and extracting the square roots, one obtains for \( a \) and \( a^+ \) the following representation

\[ N|n\rangle = \frac{1}{2} (aa^+ + a^+a)|n\rangle = n + \frac{p}{2}, \]
\[ [a, a^+] = 1 + (p-1)(-1)^n, \quad p = 0, 1, 2, \ldots \]
\[ <2n|a|2n+1> = \sqrt{2n + p}, \]
\[ <2n-1|a|2n> = \sqrt{2n} \]

(2.2)

As is seen, for \( p \) (referred to as the order of the statistics) = 1, the distinction between the odd and even states disappear and we recover the standard harmonic oscillator.

Now we study the algebra Eq.(2.1) in all generality. We follow the procedure used by Gruber and Klimyk for the Lie algebras [6]. We choose a basis \( \Omega \) for the universal enveloping algebra of the parabose oscillators as

\[ \Omega : \{a^+, a N, n, m, r, = 0, 1, 2, \ldots \} \equiv X(n, m, r), \]
\[ X(0, 0, 0) = I \]

(2.3)

where \( X(n,m,r) \) is an ordered (tensor) product of the elements \( a^+, a \) and \( N \). An element \( y \in \Omega \) is called an extremal vector for the representation \( \rho \) on \( \Omega \) if

\[ \rho(\beta)y = 0, \quad \beta = a \ or \ a^+ \ldots \]

(2.4)