SELF-DUAL MONOPOLES AND CALORONS

W. Nahm

Physikalisches Institut
der Universität Bonn
Nussallee 12, D-5300 Bonn
W. Germany

1. The ADHM construction for instantons, self-dual monopoles and calorons

The study of nonlinear partial differential equations remained outside the mainstream of mathematics, because their solution spaces seemed to be rather arbitrary and complicated. But physicists discovered that some of those equations occur naturally, and a closer study by both physicists and mathematicians reveals more and more beautiful structures.

Among them, the Yang-Mills equations in four dimensions are today the most outstanding ones. Results on general solutions are still scanty, but quite a lot is known about the more specialized solutions of the self-duality equation for Yang-Mills fields on euclidean four-manifolds. Indeed, this equation already became a valuable tool in the study of differentiable four-manifolds.

A basic step in the investigation of this equation was the ADHM construction[1)] of all instantons, i.e. of all gauge potentials in R^4 with self-dual and square integrable field strengths. The construction uses the cohomology of certain sheaves over the twistor space, which does not yet belong to the tool kit of many physicists. Thus we shall give an elementary modification of it, which also has the advantage of being easily generalizable to self-dual monopoles and calorons. We only consider the gauge group SU(n), but it is easy to specialize to the other classical Lie groups.

For the space coordinates and the covariant derivatives we use the standard quaternionic notation

$$x = q_\mu x^\mu, \tag{1}$$

$$D = q^\mu \mathcal{D}_\mu, \tag{2}$$

and we represent the quaternions by (2,2) matrices. The self-duality equation for the field strength may be written in the form

$$D^+ D = \mathcal{D}^2 \cdot 1_2 \tag{3}$$

i.e. D^+D commutes with the quaternions.

Let ψ be a matrix whose columns form an orthonormal basis of the solutions of the Weyl equations in the background of the gauge field, such that

$$D^{+}\psi = 0 \tag{4}$$

and

$$\int \psi^{+}\psi \, d^4x = 1_k \tag{5}$$

where k is the number of linearly independent square integrable solutions of eq.(4). Because of eq. (3) the adjoint equation has no solution, such that k is the index of D^+.

The projector onto the solution space of eq. (4) may be written in the form

$$\psi\psi^{+} = 1 - DGD^{+} \tag{6}$$

with

$$G = (\mathcal{D}^2)^{-1} \tag{7}$$

Because of the conformal invariance of the Weyl equation it is easy to determine the asymptotic behaviour of ψ . We write it in the form

$$\psi = \pi^{-1} x \varepsilon \sigma \alpha / r^4 + O(r^{-4}) \tag{8}$$

where α is a constant (n,2,k) matrix and

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{9}$$

transforms the $q\mu$ to their complex conjugates. Moreover σ is a matrix of orthonormal covariant constants at the sphere at infinity. We may write

$$\mathcal{D}^2 \sigma = 0 \tag{10}$$

and

$$\sigma(x_\infty) = 1_n \tag{11}$$