Recognizable Sets with Multiplicities in the Tropical Semiring

Imre Simon*
Instituto de Matemática e Estatística
Universidade de São Paulo
05508 São Paulo, SP, Brasil

Abstract

The last ten years saw the emergence of some results about recognizable subsets of a free monoid with multiplicities in the Min-Plus semiring. An interesting aspect of this theoretical body is that its discovery was motivated throughout by applications such as the finite power property, Eggan's classical star height problem and the measure of the nondeterministic complexity of finite automata. We review here these results, their applications and point out some open problems.

1 Introduction

One of the richest extensions of finite automaton theory is obtained by associating multiplicities to words, edges and states. Perhaps the most intuitive appearence of this concept is obtained by counting for every word the number of successful paths spelling it in a (nondeterministic) finite automaton. This is motivated by the formalization of ambiguity in a finite automaton and leads to the theory of recognizable subsets of a free monoid with multiplicities in the semiring of natural numbers. This theory leads, in turn, to the consideration of semigroups of matrices with coefficients in $\mathbb{N}$ and it encounters some classical results from algebra and analysis in the context of representation theory and (formal) power series (in noncommuting variables).

In finite automaton theory this approach was pioneered and vigorously pursued since the early sixties by the "French School" led by Marcel Paul

*This work was supported by FAPESP and CNPq

Schützenberger. A major step was undertaken by Samuel Eilenberg who sistematized both the formalism and the most important results in his seminal book, [12], published in 1974. In particular he explicited the machinery and this prompted the consideration of multiplicities in any semiring K. The two most important particular cases studied in Eilenberg's book are given by the boolean semiring (leading to classical finite automata) and the semiring of natural numbers (leading to the consideration of ambiguity in finite automata). More recent treatments of the subject can be found in [3,32].

In 1978 the author was led to the investigation of recognizable sets with multiplicities in another semiring, denoted $\mathcal{M}$, in [33,35]. This is just the semiring of the natural numbers extended with ∞ under the operations of taking minimums and addition. Such semiring, sometimes called the Min-Plus semiring, is important in operations research where it is used in problems of cost minimization [7]. Here, we shall call it the *tropical semiring*, a suggestion of Christian Choffrut.

Our purpose in this paper is to survey the emerging theory of recognizable subsets of a free monoid with multiplicities in the tropical semiring. We shall also point out, in our way, the applications of this theory to linguistic problems as well as to the capturing of the nondeterministic complexity of finite automata. We shall omit the proofs which can be found elsewhere.

2 The finite section and the limitedness problems

In this section we describe a problem from two different viewpoints.

Let K be a semiring and let M_nK denote the multiplicative monoid of $n \times n$ matrices with coefficients in K. Let S be a subset of M_nK. For $i, j \in [1, n]$ we define the *(i,j)-section* of S as being the set of cofficients (i, s, j), when s runs over S. A subset of K is called a *section* of S if it is an (i,j)-section for some i and j. Clearly, set S is finite if and only if every section of S is finite.

The *finite section problem* (for K) takes a finite subset X of M_nK and a pair (i,j) of indices as input. It consists of deciding whether or not the (i,j)-section of the subsemigroup of M_nK generated by X is finite.

Another, more restricted, problem is given by the *finite closure problem* (for K), which consists of deciding whether or not the subsemigroup of M_nK generated by a given finite set of matrices is finite or not. Clearly, whenever the finite section problem is decidable so is the finite closure problem. But the converse does not hold in general.