Abstract:
This paper introduces positive/negative conditional term rewriting systems, with rules of the generic form:
\[ u = v \land u' \neq v' \Rightarrow \lambda \rightarrow \rho, \]
as they often appear in algebraic specifications. We consider the algebraic semantics of such systems (viewed as sets of axioms). They do not in general have initial models; however, we show that they admit quasi-initial models, that are in some sense extremal within the class of all models. We then introduce the subclass of reducing rewrite systems, constrained by the condition: \( \lambda > \rho, u, v, u', v' \) (for some reduction ordering \( > \)). For such systems, we show that an optimal rewrite relation \( \rightarrow \) may be defined, and constructed as a "limit". We prove the total validity of an interpreter that computes the normal forms of terms for \( \rightarrow \). It is then shown that when \( \rightarrow \) is confluent, the algebra of normal forms is a quasi-initial model. We state a general result about the converse. Lastly, we present a complete critical-pair criterion à la Knuth-Bendix to check for the confluence of reducing systems.

1. Introduction: positive/negative conditional TRS

The field of term rewriting systems has seen important developments during the last decade. One crucial aspect of rewriting is that it provides a natural, operational interpretation of algebraic specifications. Strong connections are known to exist between the two domains – from the algebraic, operational and logical points of view.

Originally, term rewriting systems were limited to equational rules, i.e. formulae of the form: \( \lambda \rightarrow \rho \). These rules are not, however, expressive enough to simulate the algebraic specifications one tends to write naturally. More recently, attention has been devoted to positive conditional rules, of the generic form: \( u = v \Rightarrow \lambda \rightarrow \rho \). In this paper, we further generalize the class of rules under consideration, introducing positive/negative conditional rules of the form:
\[ u = v \land u' \neq v' \Rightarrow \lambda \rightarrow \rho. \]

These correspond exactly to the axioms used, naturally, when writing algebraic specifications. In spite of their obvious importance, this class of rules has not been studied so far, perhaps for the following reasons:

- They do not exhibit a straightforward algebraic behaviour. This is to be contrasted with the cases of equational and positive conditional systems, which have initial algebra semantics. In the positive/negative case, as shown below, the specifications do not admit, in general, an initial model. However, we show that they admit what we call quasi-initial models, having interesting algebraic properties; among them, that there exists a privileged model that precisely captures the intuition of the specifier. Moreover, under favourable circumstances, this model may be described by rewriting.
• **positive/negative** systems are at least as complicated as positive ones, about which not much was known until recently. However, we feel that recent results about positive systems have implications for positive/negative ones. In particular, we systematically use in this paper the notion of **reducing** systems, satisfying the condition:

\[ \lambda \succ \rho, \ u, v, \overline{u}, \overline{v}. \]

(for some reduction ordering \( \succ \)). Intuitively, this means that the complexity of a computation decreases monotonically along the rewrite sequences, as well as along the "recursive calls" to evaluate \( u, v, \overline{u} \) and \( \overline{v} \). Reducing systems are shown to have various interesting properties. In particular, they may be assigned a minimal semantics (which is not necessarily the case for non-reducing systems), closely connected to the algebraic notion of quasi-initial models.

Note that an inequation "\( \overline{u} \neq \overline{v} \)" cannot be replaced by an expression such as "\( -\text{Eq}(\overline{u}, \overline{v}) \)" or even "\( \text{notEq}(\overline{u}, \overline{v}) \)", for defined predicates (or boolean functions) 'Eq' or 'notEq'. This is because definitions by predicate generate semi-decidable specifications (cf. e.g. [BBTW 81], [Kaplan 82]), whereas inequations allow to define co-semi-decidable specifications.

The paper is organized as follows:
- Section 1 introduces general definitions.
- Section 2 deals with the algebraic aspects of positive/negative systems; it is shown that such specifications do not admit in general an initial model. Quasi-initial models are introduced as a natural extension of the notion of initial model, and general results about the existence and completeness of quasi-initial models are presented.
- Section 3 considers positive/negative rewriting. It is proved that in the case of reducing systems, there exists a minimal rewrite relation. This relation may be constructed as a (non-monotonic) limit, and is decidable. We show the total correctness of a universal interpreter to compute it. When the relation is known to be confluent, the normal form algebra is a quasi-initial model. A converse result is also stated, establishing a strong link between the algebraic and the operational aspects of such systems.
- Lastly, section 4 considers the confluence of reducing systems. A general critical pair theorem à la Knuth-Bendix is proved.

In sections 2 to 4, only the most interesting proofs are presented. The remaining proofs are to be found in the appendices.

We assume that the reader has basic knowledge about term rewriting systems and algebraic specifications. We refer to the papers [HO 80], [Klop 87], and to [ADJ 78], [EM 85], as general introductions to the former and to the latter fields, respectively.

Throughout this paper, a signature \( \Sigma \) is given, together with a set of variables \( X \). \( T_\Sigma \) stands for the set of ground terms on \( \Sigma \), and \( T_\Sigma(X) \) for the set of terms with variables on \( \Sigma \) and \( X \).