Generalizations of the Periodicity Theorem of Fine and Wilf

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Abstract. We provide three generalizations to the two-dimensional case of the well known periodicity theorem by Fine and Wilf [4] for strings (the one-dimensional case). The first and the second generalizations can be further extended to hold in the more general setting of Cayley graphs of groups. Weak forms of two of our results have been developed for the design of efficient algorithms for two-dimensional pattern matching [2, 3, 6].

1 Introduction

Periodicity, overlap, power, repetition. All these words could be considered as synonym of the same notion. For instance, for algebraic structures such notion is the iterated product of the same element, i.e., a power. For combinatorics on words and algorithmic applications derived from it, the notion is the “periodic appearance” of the same sequence of letters, i.e, periodicity.

M. P. Shutzenberger in [9, ch. 8] says that: ”Periodicity is an important property of words, that is often used in applications of combinatorics on words. Main results characterizing it are the Theorem of Fine and Wilf and the Theorem of the Critical Factorization...”

Indeed, apart from its intrinsic mathematical interest, periodicity of words is a very important tool for the design and analysis of algorithms on words. For instance, to the best of our knowledge, there is no optimal (sequential or parallel) string matching algorithm that does not use one of the two main theorems of periodicity either for the design or for the analysis of the algorithm. The reader is referred to [1] for a survey of such algorithms.

Despite the rich body of knowledge available for periodicity of words [9], very little was known about such notion in higher dimensions, e.g., coverings of two-dimensional space by means of a repeated pattern. Motivated by applications in low level image processing [12], namely the design of efficient two-dimensional run length compressed matching algorithms, Amir and Benson [2] started a formal characterization of the intuitive idea of what two-dimensional periodicity should be. Indeed, in their seminal paper [2], they came up with a definition of periodicity for two-dimensional patterns, provided a classification of such patterns in four periodicity classes and devised efficient algorithms that decide in which class a given input pattern falls into. Such combinatorial study has been successfully applied by Amir and Benson to the design of efficient two-dimensional pattern matching algorithms on compressed data [2] and by Amir, Benson and Farach [3] to make progress towards
a two-dimensional pattern matching algorithm that runs in linear time, independent of the alphabet size. Motivated by this second problem, Galil and Park have brought to light additional properties of two-dimensional periodicity and have applied them to obtain the first truly alphabet independent two-dimensional pattern matching algorithm. We point out that the two-dimensional periodicity studied in [2, 6, 7] is a natural extension of periodicity for words and that the results they prove about it can be regarded as the two-dimensional analog of the Periodicity Lemma for words [10]. It is well known that the Periodicity Lemma is a weak form of the Fine and Wilf Theorem [4].

Recently, Regnier and Rostami [13] have provided a framework for the study of d-dimensional periodicities. Based on it, they identify $2^{d-1} + 1$ classes of periodicities for a d-dimensional pattern. For $d = 2$, their classification is a refinement of the one in [2, 6]. They also proved results that can be considered d-dimensional generalizations of the Periodicity Lemma. Algorithmic applications of such theory are claimed in [11].

In this paper we introduce the notion of periodicity on Cayley graphs of groups. Roughly speaking, that corresponds to placing the notion of higher dimensional periodicity discussed in [2, 6, 13] into a very general setting. Moreover, in that setting, we prove three generalizations of the theorem of Fine and Wilf for words. The first and the second generalizations hold in Cayley graphs of groups while the third holds only in the case in which we restrict the group to be $\mathbb{Z}^2$. The relationship between the results previously known for d-dimensional periodicity [2, 6, 13] and ours is the same as the one between the Periodicity Lemma and the Theorem of Fine and Wilf for words. Indeed, the Periodicity Lemma provides conditions for a word to be periodic while the Theorem of Fine and Wilf provides the tightest possible conditions for a word to be periodic. Indeed, weak forms of the second and the third of our generalizations have been obtained by Amir and Benson [2] and Galil and Park [7] and used in the design and analysis of their algorithms [2, 7].

The remainder of this abstract is organized as follows. In the second section we give some basic notations and discuss some preliminaries. In the third section we states three theorems and we show that each of them is a generalization of the Theorem of Fine and Wilf; proofs of these results are given in the fourth section. The last section presents conclusions and some open problems.

2 Preliminaries

For any notation not explicitly defined in this paper we refer to [9] and to [10].

We present now few group-theoretic preliminaries.

Let $G$ be an additive group. Given any subgroup $L$ of $G$, for every $g \in G$, $g + L = \{g + l \mid l \in L\}$ is the right coset of $L$ containing $g$.

The set of right cosets of $L$ is a partition of $G$, i.e., each element of $G$ belongs to exactly one right coset. The cardinality of the set of right cosets is denoted by $i(L)$; we remark that $i(L)$ is not necessarily a finite number. A transversal $T_L$ is a subset of $G$ containing exactly one element from every right coset. It is clear that $\text{Card}(T_L) = i(L)$. 