Multi-Dimensional Signal Processing Using an Algebraically Extended Signal Representation*

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Abstract. Many concepts that are used in multi-dimensional signal processing are derived from one-dimensional signal processing. As a consequence, they are only suited to multi-dimensional signals which are intrinsically one-dimensional. We claim that this restriction is due to the restricted algebraic frame used in signal processing, especially to the use of the complex numbers in the frequency domain. We propose a generalization of the two-dimensional Fourier transform which yields a quaternionic signal representation. We call this transform quaternionic Fourier transform (QFT). Based on the QFT, we generalize the conceptions of the analytic signal, Gabor filters, instantaneous and local phase to two dimensions in a novel way which is intrinsically two-dimensional. Experimental results are presented.

1 Introduction

Realizations of autonomous technical systems which are designed on principles of the perception-action-cycle (PAC) are supposed to act in the real world which can be described as taking place in a four-dimensional Euclidean space-time. Therefore, a PAC-system has to be able to percept events and to organize processes in such a world.

Focusing on the perceptional part of a PAC system we face some serious shortcomings in low-level processing of multi-dimensional signals. These shortcomings have been recognized for a long time but by now are not solved satisfactorily. In the authors opinion the root of the problems seems to lie in the restricted algebraical embedding of multi-dimensional signal-processing which has not yet been recognized. The algebraical embedding of signal processing is meant to be the choice of algebra in which a signal is represented in the frequency domain. Usually this role is played by the algebra of complex numbers but we will show that it is useful to apply algebras of higher dimensions here.

Most of the valuable tools that have been developed in one-dimensional signal processing are nowadays used in multi-dimensional signal processing but in a way which leaves them intrinsically one-dimensional.

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One example for this is the concept of local phase. The local phase of a one-dimensional signal can be estimated by applying a quadrature filter — e.g. a complex Gabor filter — and evaluating the argument of the resulting complex filter response. This concept is usually generalized to two dimensions by defining the two-dimensional Gabor filters as the Gaussian windowed basis function of the Fourier transform for some frequency $u$. Again, we get as the local phase the argument of the complex filter response and we get different values for different orientations of the Gabor filter. Granlund [7] defines an $(n+1)$-dimensional phase vector for $n$-dimensional signals, consisting of the real phase and the directional vector of the chosen orientation.

In one dimension the local phase yields information about the local structure of the signal. In two dimensions the variety of possible local structures is much higher than in one dimension and so we cannot hope to characterize the local image structure using only one real number. Looking for a concept which yields a higher-dimensional value for the local phase we find that the main restriction of the phase dimension lies in the fact that the responses of the Gabor filters are complex-valued. Thus, we will study filters with responses which are elements of a higher-dimensional algebra than the complex numbers. We will show that a generalization to quaternion-valued filters is possible in two dimensions. A short review on quaternions will be given in the following section.

One-dimensional Gabor filters are based on the Fourier transform. Therefore we will extend the two-dimensional Fourier transform in such a way that it yields a quaternion-valued representation in the frequency domain. We call this transform quaternionic Fourier transform (QFT). We will demonstrate the shift theorem in the case of the QFT, analyze the symmetry properties of the QFT and show its relation to the Fourier transform and to the Hartley transform.

In order to define the instantaneous phase of a two-dimensional signal we will introduce the quaternionic analytic signal of a two-dimensional signal via the QFT. Finally we introduce the quaternionic Gabor filters based on the QFT and the two-dimensional local phase and demonstrate some experimental results.

## 2 Quaternions

As motivated in the introduction we need as the range of a generalized two-dimensional Fourier transform an algebra whose dimension is higher than the dimension of the algebra of complex numbers. In the following we will use the four-dimensional $\mathbb{R}$-algebra

$$\mathbb{H} = \{ q = a + bi + cj + dk \mid a, b, c, d \in \mathbb{R} \} ,$$  \hspace{1cm} (1)

where $i$, $j$ and $k$ obey the following multiplication rules:

$$i^2 = j^2 = -1, \quad k = ij = -ji \quad \Rightarrow \quad k^2 = -1 .$$  \hspace{1cm} (2)