Optimal Circular Arc Representations

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Abstract. We investigate some properties of minimal interval and circular arc representations and give several optimal parallel recognition and construction algorithms. We show that, among other things, given an $s \times t$ interval or circular arc representation matrix,
- deciding if the representation is minimal can be done in $O(\log s)$ time with $O(st/\log s)$ EREW PRAM processors, or in $O(1)$ time with $O(st)$ Common CRCW PRAM processors;
- constructing a minimum interval representation can be done in $O(\log(st))$ time with $O(st/\log(st))$ EREW PRAM processors, or in $O(\log t/\log \log t)$ time with $O(st \log \log t/\log t)$ Common CRCW PRAM processors, or in $O(1)$ time with $O(st)$ BSR processors.

1 Preliminaries

Circular arc graphs are well-known intersection graphs and properly contain interval graphs. Benzer [4] showed overlap data involving fragments of a certain gene could be modeled by intervals. This finding confirmed the hypothesis that DNA has a linear structure within genes and helped him win a Nobel Prize. Circular arc graphs also find applications in some other areas such as register allocation. The best way to allocate registers corresponds to an optimal coloring of an interference graph which is often a circular arc graph or even an interval graph (see, e.g., [16]). Many algorithms on circular arc graphs work on circular arc representations (see, e.g., [15]), which can be constructed from circular arc graphs (see, e.g., [9]). In this paper, we study the properties of minimal interval and circular arc representations and present some efficient recognition and construction algorithms.

Given a family $S$ of nonempty sets, the intersection graph $G$ has vertices corresponding to the sets in $S$ and two distinct vertices of $G$ are adjacent iff the corresponding sets in $S$ intersect. $S$ is called an intersection representation (IR) for $G$. If $S$ is a family of arcs on a circle, $G$ is called a circular arc graph. If, in addition, the family of arcs satisfies the Helly property (i.e., if several arcs mutually intersect, then the intersection of these arcs is nonempty), $G$ is called a Helly circular arc graph (a.k.a. $\Theta$ circular arc graph). $G$ is called an interval graph if $S$ is a family of intervals on a real line.

In this paper, we often use a pair of the two endpoints of an arc to denote a closed arc. If we move along an arc in the clockwise direction, the last point on the arc is clockwise endpoint. The other endpoint is counterclockwise endpoint. If we use $[l_0, l_1]$ to denote an arc, $l_1$ and $l_0$ represent clockwise and counterclockwise endpoints, respectively.

The aforementioned classes of graphs can also be defined as intersection graphs on some discrete objects. Take the circular arc graphs for example. Let $D$ be a
circularly ordered set (such as points on a circle). A circular arc of $D$ is defined as any set of contiguous elements in $D$. Let $S$ be a set of circular arcs on $D$. The intersection graph $G$ of $S$ is a circular arc graph and the pair $(D, S)$ is a circular arc representation. Two IRs $(D_1, S_1)$ and $(D_2, S_2)$ are said to be equivalent if there exists a one-to-one onto function $f: S_1 \rightarrow S_2$ such that $x$ and $y$ in $S_1$ intersect iff $f(x)$ and $f(y)$ in $S_2$ intersect. An IR $(D, S)$ is said to be minimal if there does not exist an equivalent IR $(D', S')$, where $D' \subseteq D$. An IR $(D, S)$ is said to be minimum if, for any other equivalent IR $(D', S')$, $|D'| \geq |D|$. We call $|D|$ the size of the IR $(D, S)$. An element, say $d$, in $D$ is called an intersection point (IP) if there exist two elements (not necessarily distinct), say $s_1$ and $s_2$, in $S$ such that $s_1 \cap s_2 = \{d\}$. An IR, say $(D, S)$, is often denoted by a $|S| \times |D|$ $(0, 1)$-matrix. A row, say $R$, of the matrix corresponds to an element in $S$. $R(i) = 1$ iff the $i$th element of $D$ is contained in the element of $S$. We simply refer to the matrix as an IR if no confusion arises.

Suppose $D_1 = \{\bullet, \bigcirc, \bigtriangledown, \triangle, \bigstar\}$, and $S_1 = \{\{\bullet, \bigcirc\}, \{\bigcirc, \bigtriangledown\}, \{\bigtriangleup, \bigstar\}, \{\bigstar\}\}$. The corresponding matrix is $M_1$. Let $D_2 = \{J, Q, K, A\}$, and $S_2 = \{\{J\}, \{J, Q, K\}, \{Q, K, A\}, \{A\}\}$. Then the corresponding matrix is $M_2$. It is easy to verify these two interval representations are equivalent. The minimal interval representation is $M_3$, which can be obtained by deleting a column from $M_1$ or $M_2$.

$$M_1 = \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad M_2 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad M_3 = \begin{bmatrix}
1 & 0 & 0 \\
1 & 1 \\
0 & 1 \\
0 & 0 & 1
\end{bmatrix}$$

The computation models employed in this paper are more or less standard. One model used is the well known PRAM. Some of our algorithms are implemented on EREW PRAM. Some other algorithms are designed for the Common CRCW PRAM. Also mentioned is a stronger submodel called Priority CRCW PRAM. A more powerful model known as Broadcasting with Selective Reduction (BSR) introduced in Akl and Guenther [2] is also used here.

A PRAM algorithm is said to be work-optimal if the processor-time product matches the time lower bound of the sequential algorithm. We say an algorithm is time-optimal if its time bound matches the lower bound on the corresponding model.

In designing PRAM algorithms, we often use the following result, usually attributed to Brent [5], to obtain the best tradeoff between the time and processor bounds.

**Theorem 1.** If a problem can be solved in $O(T)$ time with $O(P)$ PRAM processors, the problem can also be solved in $O(TP/P')$ time with $O(P')$ PRAM processors, for $P' < P$.

Cook, Dwork and Reischuk [12] established the following lower bound.

**Theorem 2.** Computing the OR of $n$ bits requires at least $\Omega(\log n)$ time on exclusive-write machines.