\textbf{\textit{\varepsilon-Approximation of Differential Inclusions}}\footnote{Research supported by the California PATH program and by the National Science Foundation under grant ECS9417370.}

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\textbf{Abstract.} For a Lipschitz differential inclusion $\dot{x} \in f(x)$, we give a method to compute an arbitrarily close approximation of $\operatorname{Reach}_f(X_0, t)$ — the set of states reached after time $t$ starting from an initial set $X_0$. For a differential inclusion $\dot{x} \in f(x)$, and any $\varepsilon > 0$, we define a finite \textit{sample graph} $A^\varepsilon$. Every trajectory $\phi$ of the differential inclusion $\dot{x} \in f(x)$ is also a "trajectory" in $A^\varepsilon$. And every "trajectory" $\eta$ of $A^\varepsilon$ has the property that $\text{dist}(\dot{\eta}(t), f(\eta(t))) \leq \varepsilon$. Using this, we can compute the \textit{\varepsilon-invariant} sets of the differential inclusion — the sets that remain invariant under \varepsilon-perturbations in $f$.

\section{Introduction}

A dynamical system $\dot{x} \in f(x)$ describes the flow of points in the space. Associated with a dynamical system are several interesting concepts: from an \textit{invariant set}, points cannot escape; and a recurrent set is visited infinitely often. For the controlled system $\dot{x} = f(x, u)$, the question of whether there is a control $u \in U$ to steer the system from an initial state $x_0$ to a final state $x_f$ is fundamental.

We approach the subject from the viewpoint of applications and an interest in computational methods. For the differential inclusion $\dot{x} \in f(x)$, we want to compute the invariant sets and the recurrent sets. For the controlled differential equation $\dot{x} = f(x, u)$, we want to determine the control $u \in U$ which steers the system from an initial state $x_0$ to a final state $x_f$. And we want to determine the reach set $\operatorname{Reach}_f(X_0, [0,t])$ — the set of states that can be reached from the initial set of states $X_0$ within time $t$.

In this paper, we propose a computational approach to solve some of these problems. For a Lipschitz differential inclusion $\dot{x} \in f(x)$ with initial set $X_0$, we propose a polyhedral method to obtain an arbitrary close approximation of $\operatorname{Reach}_f(X_0, [0,t])$. For a differential inclusion $\dot{x} \in f(x)$, and any $\varepsilon > 0$, we construct a finite \textit{sample graph} $A^\varepsilon$ which has the property that every trajectory $\phi$ of $\dot{x} \in f(x)$ is also a "trajectory" in the graph $A^\varepsilon$. And every "trajectory" $\eta$ of the finite graph $A^\varepsilon$ has the property that $\text{dist}(\dot{\eta}(t), f(\eta(t))) \leq \varepsilon$. Since $A^\varepsilon$ is
a finite graph, it can be analyzed using graph theoretic techniques. Using the finite graph $A^\epsilon$, we can compute the $\epsilon$-invariant sets of $\dot{x} \in f(x) = \{ z \in f(x) \}$ — the sets which remain invariant under $\epsilon$-perturbations in $f$.

In Section 2, we introduce our notation, and define the basic terms. In Section 3, we conservatively approximate the differential inclusion by a piecewise constant inclusion, and obtain an approximation of $Reach_f(X_0, [0, t])$. In Section 4, we obtain a finite graph $A^\epsilon$ from the differential inclusion $\dot{x} \in f(x)$, and use it to determine the properties of the differential inclusion. In Section 5, we discuss the application of techniques from Sections 3 and 4 to computing the $\epsilon$-invariant sets of differential inclusions. In Section 6, we apply these methods to compute the invariant sets for two examples: a pendulum moving in the vertical plane, and the Lorenz equations. We also discuss procedures to improve the efficiency of our methods. Section 7 is the conclusion.

2 Preliminaries

Notation

$\mathbb{R}$ is the set of reals and $\mathbb{Z}$ is the set of integers. $B = \{ x : |x| \leq 1 \}$ is the unit ball. For sets $U, V \subseteq \mathbb{R}^n$, $U + V = \{ u + v | u \in U \text{ and } v \in V \}$ and for $\alpha \in \mathbb{R}$, $\alpha U = \{ \alpha u | u \in U \}$. For $\delta > 0$, $B_\delta(x)$ is the $\delta$-ball centered at $x$, i.e., $B_\delta(x) = \{ y : |y - x| \leq \delta \}$. For $X \subseteq \mathbb{R}^n$, $X_\epsilon = X + \epsilon B$.

For $x \in \mathbb{R}^n$, and $Y \subseteq \mathbb{R}^n$, the distance $\text{dist}(x, Y) = \inf \{|x - y| : y \in Y\}$. For two sets $X,Y \subseteq \mathbb{R}^n$, the Hausdorff distance is $\text{dist}(X,Y) = \inf \{|r : X \subseteq Y + rB \text{ and } Y \subseteq X + rB\}$. Notice, that if $\text{dist}(X,Y) \leq \epsilon$, then for any $x \in X$, $\text{dist}(x, Y) \leq \epsilon$. For $X \subseteq \mathbb{R}^n$, $\text{cl}(X)$ is the closure of $X$, and $\text{co}(X)$ is the smallest closed convex set containing $X$. For $X, Z \subseteq \mathbb{R}^n$, the restriction of $X$ to $Z$ is $X|_Z = X \cap Z$. For a set $J$, the complement of $J$ is $J^c$. For sets $X$ and $Y$, the difference $X \setminus Y = \{ z | z \in X \text{ and } z \not\in Y \}$.

A set-valued (multi-valued) function is $f : \mathbb{R}^n \to \mathbb{R}^n$ where $f(x) \subseteq \mathbb{R}^n$. For a set-valued $f : \mathbb{R}^n \to \mathbb{R}^n$, the set-valued function $f_\epsilon : \mathbb{R}^n \to \mathbb{R}^n$ is given by $f_\epsilon(x) = f(x) + \epsilon B$. For $Z \subseteq \mathbb{R}^n$, $f(Z) = \bigcup_{x \in Z} f(x)$. We assume the infinity norm on $\mathbb{R}^n$ (i.e., $|x| = \max\{|x_1|, \ldots, |x_n|\}$).

Differential Inclusions

A differential inclusion is written as $\dot{x} \in f(x)$ where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a set-valued function. Differential inclusions can be used to model disturbances and uncertainties in the system. A differential equation $\dot{x} = f(x, u)$, where $u \in U$ is control or disturbance can be studied as the differential inclusion $\dot{x} \in g(x)$ where $g(x) = \{ f(x, u) | u \in U \}$. The differential inclusion $\dot{x} \in g(x)$ captures every possible behaviour of $f$.

We say a differential inclusion $\dot{x} \in f(x)$ is Lipschitz with Lipschitz constant $k$ provided $\text{dist}(f(x_1), f(x_2)) \leq k|x_2 - x_1|$. A trajectory $\phi : \mathbb{R} \to \mathbb{R}^n$ is a solution of $\dot{x} \in f(x)$ provided $\phi(t) \in f(\phi(t))$ a.e. We say $f$ is convex-valued when $f(x)$ is convex for every $x$. 