MULTIDIMENSIONAL CONSTANT LINEAR SYSTEMS

by Ulrich Oberst, Institut für Mathematik, Universität Innsbruck

This talk is based on my paper "Multidimensional constant linear systems" which recently appeared in Acta Appl. Math. 20(1990), 1-175, and explains some of the system ingredients and results which are discussed in this paper, in particular signals, linear systems of partial differential or difference equations with constant coefficients, the behaviour of a system, input-output structures and the associated transfer matrices of systems, the discrete Cauchy (or initial value) problem and the transfer operator. The proofs, many further results and detailed references can be found in [loc.cit.].

SIGNALS

The state, input and output of a system are usually described by real-valued functions which are customarily called signals in system theory. Let $\mathcal{X}$ denote a real vector space of functions (signals). For simplicity I consider only two typical cases in this talk. Let $r$ be a positive integer.

Signals in the continuous case: Let $\mathcal{X} := C^\infty(\mathbb{R}^r)$ denote the $\mathbb{R}$-vector space of infinitely often differentiable real-valued functions $a = a(t_1, \ldots, t_r)$ of $r$ real variables $t_1, \ldots, t_r$. Another admissible signal space, more suitable for many applications, is the space $\mathcal{D}'(\mathbb{R}^r)$ of $r$-dimensional distributions.

Signals in the discrete case: Let $\mathcal{X} := \mathbb{R}^{N^r}$ or $\mathcal{X} := \mathbb{R}^{Z^r}$ denote the $\mathbb{R}$-vector space of multindexed sequences. An element $z$ of $\mathcal{X} = \mathbb{R}^{Z^r}$ is a real-valued function $a = (a(n); n \in \mathbb{Z}^r) = (a(n_1, \ldots, n_r); n_i \in \mathbb{Z}, i = 1, \ldots, r) : \mathbb{Z}^r \to \mathbb{R}, n \mapsto a(n)$, on the $r$-dimensional integral lattice $\mathbb{Z}^r \subset \mathbb{R}^r$. The lattice $\mathbb{N}^r$ is just the positive quadrant in $\mathbb{Z}^r$.

For $r = 1$ the independent variable $t \in \mathbb{R}$ resp. $n \in \mathbb{Z}$ is usually interpreted as the continuous resp. discrete time. The corresponding systems (see below) are called time-systems or one-dimensional or "classical". For $r \geq 1$ one talks about multidimensional ($r$-dimensional, $r$-d-) systems. The independent variables $t_1, \ldots, t_r$ resp. $n_1, \ldots, n_r$ are interpreted as temporal and spatial coordi-
nates or, more generally, as the coordinates of a phase space of arbitrary dimension. The systems are called multivariable if they are described by vector functions \( w \in \mathbb{R}^1, l > 1 \).

**Example (image processing):** For \( r = 2 \) the lattice points \( n = (n_1, n_2) \in \mathbb{Z}^2 \) are interpreted as the discrete coordinates of an image. The value of \( a(n) \) is the light intensity at the point \( n \). In a completely digital model one replaces the field \( \mathbb{R} \) of real numbers by a finite field (see [loc.cit.]).

**LINEAR EQUATIONS AND SYSTEMS**

**Standard examples (for \( r = 1 \)):**
(i) In the continuous case consider the ordinary second order, linear differential equation \( y'' + a_1 y' + a_0 y = v \) in \( \mathbb{C}^\infty(\mathbb{R}) \) where \( v \) is a given \( \mathbb{C}^\infty \) function and \( a_1 \) and \( a_0 \) are real constants. With the derivative operator \( s y := y' := dy/dt \) the differential equation gets the operator form \( R y := R(s)y = v \) with the characteristic polynomial \( R := R(s) := s^2 + a_1 s + a_0 \in \mathcal{R}[s] \).

(ii) The customary state space equations
\[
\dot{x} = Ax + Bu \quad \text{with } x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{C}^\infty(\mathbb{R})^n, u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}, \quad A \in \mathbb{R}^{n,n}, \ B \in \mathbb{R}^{n,m},
\]

can equally be written in the operator form \( (sI_n - A)y = Bu \).

(iii) In the discrete case consider the famous Fibonacci equation
\[
y(n+2) = y(n+1) + y(n), \quad n \geq 0, \quad \text{or} \quad (s^2 - s - 1)y = 0
\]
for the population dynamics of female rabbits with the left shift operator \( s \),
\[
sy := (y(1), y(2), \ldots) \in \mathbb{R}^N \text{ for the sequence } y = (y(0), y(1), y(2), \ldots) \in \mathbb{R}^N.
\]

Analogous operator equations can be formed in the multidimensional situation. For this purpose let \( \mathcal{R}[s] := \mathcal{R}[s_1, \ldots, s_r] \) denote the \( \mathbb{R} \)-algebra of real polynomials \( \mathcal{R} \) in \( r \) indeterminates \( s_1, \ldots, s_r \). A \( \mathbb{R} \)-basis of \( \mathcal{R}[s] \) is formed by the monomials
\[
s^m := s_1^{m(1)} s_2^{m(2)} \cdots s_r^{m(r)} \quad \text{where } m := (m(1), \ldots, m(r)) \in \mathbb{N}^r.
\]

A polynomial is then a finite linear combination
\[
R := \sum_{m \in \mathbb{N}^r} R(m) s^m, \quad R(m) \in \mathbb{R}, \quad R(m) = 0 \text{ for almost all } m.
\]

As in the one-dimensional examples above one defines a scalar multiplication
\[
\mathcal{R}[s] \times \mathbb{R} \to \mathcal{R}, \ (R,a) \mapsto Ra := \sum_{m \in \mathbb{N}^r} R(m) s^m a,
\]