INTEGRABLE AND STOCHASTIC BEHAVIOUR IN DYNAMICAL ASTRONOMY

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ABSTRACT

Some problems of Stellar Dynamics and Celestial Mechanics are presented, where integrability and stochasticity play a role. Such problems are: 1) the motions of stars in the meridian plane of an axisymmetric galaxy, 2) the motions in the plane of symmetry of a spiral galaxy, 3) the escapes of stars to infinity, 4) the bifurcations of families of periodic orbits, 5) Lynden-Bell's statistics in collapsing systems, 6) the general three-body problem, and 7) the applicability of Arnold's diffusion.

I. INTRODUCTION

Dynamical Astronomy is divided into Stellar Dynamics and Celestial Mechanics. The first deals with stellar systems composed of large numbers of stars, while the second deals with systems of few bodies, with the main emphasis on our planetary system, where the dominant role is played by the sun.

The notions of integrability and stochasticity play an important role in many problems of Stellar Dynamics and Celestial Mechanics and I will give some examples from both fields, mentioning also some of the most recent results.

A large part of Stellar Dynamics is devoted to collisionless problems, i.e. problems in which the time of relaxation is long. In fact in a galaxy of $10^{11}$ stars the time of relaxation is of the order of $10^{13}$ years, and this is not only much larger than the period of galactic rotation, which is equal to a few $10^8$ years, but also larger than the age of the universe itself. In other words relaxation, due to the grainy character of the galaxy (as distinguished from a smooth average potential), is small.

The relaxation time of smaller systems (stellar clusters) is much shorter, but still longer than the orbital period. Therefore in many cases we can ignore the grainy character of a stellar system and consider the orbits of stars in the average, smooth, field.
II. STELLAR DYNAMICS

a) The third integral

The simplest type of problems in Stellar Dynamics deals with orbits of stars in a given potential that represents a model of a galaxy, or of a cluster. For example we may be interested to know the orbit of a star of known age in order to find its place of origin.

In general, however, we are not interested in particular orbits, but in the statistical behaviour of the stars in a stellar system. In particular we are interested in the distribution of stars in space and in the distribution of their velocities.

In collisionless dynamics the density of stars in phase space, \( f \), (\( f \) is called also "distribution function"), is a constant of motion. In other words the stars move in phase space \( \mu \) as an incompressible fluid. Thus if we have, say, \( dN \) stars in a phase element, these stars move around in time, but the volume, \( dV \), that they occupy is always equal, and thus the density, \( f \), remains the same.

However the shape of \( dV \) changes, and in general after some time this element becomes a very complicated long string that approaches every point in the available phase space. In such a case it would be useless to speak about the density \( f \) at a particular point, but we should consider only some average, or "coarse grained", density, \( \bar{f} \), as meaningful.

But what is meant by "available phase space?" Of course if the field is time independent, the energy, \( E \), of each star does not change. Also if the system has axial symmetry one component, \( J \), of the angular momentum, does not change. Thus \( \bar{f} \) is a function of the integrals \( E \) and \( J \). That means that the stars do not go everywhere in phase space, but stay on the "surface" of constant \( E \) and \( J \). The question now arises whether they cover that "surface" in an ergodic, or stochastic, way, or whether there are other, further restrictions.

Twenty years ago the answer would have been that no other useful integrals exist besides the energy and the angular momentum. This was based partly on some general ideas about ergodicity, but mainly on the well-known theorem of Poincaré about the non-existence of other analytic integrals of motion.

Thus it came as a surprise when some of the first applications of