On the Separable-Homogeneous Decomposition of Graphs

Extended Abstract

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Abstract. We introduce a new decomposition scheme for arbitrary graphs which extends both the well-known modular and the homogeneous decomposition. It is based on a previously known structure theorem which decomposes a graph into its $P_4$-connected components and on a new decomposition theorem for $P_4$-connected graphs. As a final result we obtain a tree representation for arbitrary graphs which is unique up to isomorphism.

1 Introduction

It is a time-honored paradigm to model problems in communications, VLSI design, database design, network protocol design, and other areas of computer science and engineering, by graphs in the hope that the resulting graph problems can be solved fast. A powerful tool for obtaining efficient solutions to graph problems is *divide-and-conquer*, one of whose incarnations is *graph decomposition*.

In turn, an increasingly appealing approach to graph decomposition involves associating with the graph at hand $G$ a rooted tree $T(G)$ whose leaves are subgraphs of $G$ (e.g. vertices, edges, cliques, stable sets, cutsets) and whose internal nodes correspond to certain prescribed graph operations. In applications, it is most desirable that the corresponding tree representation be *unique* and that it be obtained *efficiently*, in time polynomial in the size of the graph $G$.

Tree representations satisfying these conditions are important, in particular, to solve the graph isomorphism problem. If the leaves of $T(G)$ can be tested for isomorphism in polynomial time, then the graph isomorphism problem can be solved efficiently for $G$ since it reduces to labeled tree isomorphism. Unique tree representations have been obtained for several classes of graphs, among others for cographs \([4]\), hook-up graphs \([10]\), transitive series-parallel digraphs \([11]\), interval

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graphs [3], \(P_4\)-reducible graphs [6], \(P_4\)-extendible graphs [7], \(P_4\)-sparse graphs [8] and \((q, t)\)-graphs [1].

The modular decomposition, also known as substitution decomposition, is a well-investigated type of graph decomposition which has applications in many areas of discrete mathematics. It allows to solve efficiently a wide class of combinatorial optimization problems. For a comprehensive review see [14] and [15]. Very recently, the modular decomposition has been extended to the homogeneous decomposition [9]. The main contribution of this paper is to introduce a new type of decomposition which extends the modular and the homogeneous decomposition in the sense that it goes further in decomposing graphs which are prime with respect to both decompositions.

Our decomposition scheme is based, on the one hand, on a known structure theorem for arbitrary graphs [9] which decomposes a graph into its \(P_4\)-connected components and, on the other hand, on a new decomposition theorem for \(P_4\)-connected graphs. As with the modular and the homogeneous decomposition, we obtain a tree representation for arbitrary graphs which is unique up to isomorphism and which can be obtained in polynomial time.

The concept of \(P_4\)-connectedness has been introduced by B. Jamison and S. Olariu in [9] as a generalization of the usual connectedness of graphs. By means of the above-mentioned structure theorem, it suggests a unique tree representation for arbitrary graphs. The leaves of this tree are the \(P_4\)-connected components of the graph. A detailed study of the structure of \(P_4\)-connected graphs will allow us to further decompose the leaves and to establish a unique tree representation for the \(P_4\)-connected components.

The paper is organized as follows. Section 2 presents the terminology and summarizes previous results about \(P_4\)-connectedness. In Section 3 we introduce an extension procedure that, starting with a \(P_4\) in an arbitrary graph, tries to add one vertex after the other in such a way that the graph is \(P_4\)-connected in each step. We characterize the graphs for which this procedure can be applied successfully. In Section 4 we investigate the concept of separable-homogeneous sets, which is one of the main ingredients in our decomposition scheme, and present the decomposition theorem for \(P_4\)-connected graphs. Section 5 describes the separable-homogeneous decomposition along with the associated tree representation.

2 Terminology and previous results

All graphs in this paper are finite, with no loops nor multiple edges. In addition to standard graph-theoretical terminology, compatible with [2], we need some new terms that we are about to define.

Let \(G = (V, E)\) be a graph with vertex-set \(V\) and edge-set \(E\). For a vertex \(v\) of \(G\), \(N(v)\) denotes the set of all neighbors of \(v\). If \(U \subseteq V\) then \(G(U)\) stands for the graph induced by \(U\). Occasionally, to simplify the exposition, we shall blur the distinction between sets of vertices and the subgraphs they induce, using the same notation for both. \(E(U)\) denotes the set of all edges joining vertices from